PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On Rotational Tessellations and Copses

J. C. P. Miller

Phil. Trans. R. Soc. Lond. A 1980 **293**, 599-641
doi: 10.1098/rsta.1980.0005

ON ROTATIONAL TESSELLATIONS AND COPSES

By J. C. P. MILLER
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Silver Street, Cambridge CB3 9EW, U.K.

(*Communicated by J. W. S. Cassels, F.R.S. – Received* 19 *July* 1978)

[Plates 1 and 2]

CONTENTS

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

This paper is a self-contained sequal to Miller (1970), entitled 'Periodic forests of stunted trees'. It is concerned with 'Copses' and 'Tessellations' based on an infinite background of *nodes* at the vertices of a plane tessellation of unit equilateral triangles, forming either a finite larger equilateral triangle for a copse, or an infinite doubly periodic tessellation otherwise. In any such background the unit triangles form two sets, of opposing orientations. We label the nodes individually with a 1 (called *live* nodes) or a 0 (called *vacant* nodes) in such a way that for the nodes on each triangle of one set with the same orientation the sum of the labels equals 0 (mod 2); the sum round unit triangles of the other orientation is not restricted. The tessellations are obtained by joining by an edge every pair of adjacent live nodes.

The purpose of the paper is to study which copses and tessellations exist, and to enumerate them, and to show how they may be constructed and listed. In Miller (1970) this was done for forests, slightly different from tessellations but with an identical theoretical approach. In the present paper we are particularly interested in copses and tessellations with rotational symmetry about each of a lattice of symmetry centres, either with or without reflexive symmetry as well.

A copse is determined by a particular vector of node labels along one of its edges: the symmetry studied in the present paper corresponds to having identical vectors along all three edges of the copse. We find a basis for 'permissible vectors' yielding such symmetry for each size of copse. A tessellation is determined by an infinite vector of labels – periodic in this investigation – along a straight line of adjacent nodes. This in turn is generated as a series of coefficients by a 'generating fraction' of which the denominator is a 'generating polynomial' of finite degree depending on the period.

The vector defining a copse of size $k+1$, and the minimum polynomial of degree k, generating a tessellation of period n (which will be such that $n|2^k-1$) turn out to be eigen-vectors (respectively row- and column-vectors) of the same *Pascal matrix* (consisting of a triangle of binomial coefficients, modulo 2, together with a right bottom half all zeros). These are fully studied.

Finally it is known that the product of *all* irreducible polynomials with coefficients in GF(2) and of degree dividing k is just $t^{2^k}+t$. It is shown in this paper that, in a similar fashion, all (suitably-defined) 'primary' *reflexive* polynomials of degree dividing $2k$, themselves divide $t^{2^k+1}+1$, and that all 'primary' *rotational* polynomials of degree dividing $3k$ in a similar way divide either $t^{2^k+1}+t^{2^k}+1$ or $t^{2^k+1}+t+1$. It is also established that the 'primary' polynomials of each type, i.e. *all*, *reflexive*, *rotational* and *triangular* (both rotational and reflexive) each have the same enumerating function for the respective degrees k, $2k$, $3k$, and $6k$. We also find that there is only one irreducible triangular polynomial, namely t^2+t+1.

The various types of copses and polynomials have been enumerated for a number of values of k or n, and likewise rotational and triangular tessellations for those values of n for which they exist. A substantial selection of these tables is given in the paper. A very large number of diagrams have been drawn, and a substantial, and I hope representative, selection reproduced herein.

1. Introduction

In Miller (1970) I have given definitions, mathematical properties, and enumerations of *Periodic forests of stunted trees* and of related tessellations consisting of equilateral triangles and alternate-sided hexagons. In this paper I am interested primarily in such periodic tessellations having triangular symmetry, that is the rotational symmetry of an equilateral triangle. It turns out that this study needs a full investigation of *copses*. based on backgrounds of finite equilateral triangles of nodes, both with and without triangular symmetry; this is therefore included.

In order that the present paper may be reasonably self-contained, I give first in §2 a condensed summary of the definitions and main general properties of these periodic tessellations, leaving the reader to refer to the earlier papers, listed in the reference section, for further information and proofs.

In §3 I consider the symmetry and enumeration of copses, and in §4 there is a discussion of the Pascal matrix, and its eigensystem, with, in §5, properties and enumeration of the polynomials and sequences formed from its row- and column-eigenvectors.

In §6 I discuss the symmetry of tessellations, and in §7 the construction and in §8, the enumeration of rotational tessellations from the rotational copses contained within them.

2. Definitions and principal properties

2.1. *The background of nodes and its generation*

We consider a *background* of *nodes* forming a plane equilateral triangular lattice. We choose any node as origin of coordinates, and two axes, of r and of s, through the origin along lines of nodes at minimum (unit) spacing, and at 60° to one another. The nodes then have integer coordinates $[r, s]$; if we need Cartesian coordinates, we will take the r-axis as x-axis, and the y-axis perpendicular to it in the direction of increasing s – this gives coordinates $(r+\frac{1}{2}s, \frac{1}{2}\sqrt{3}s)$ for the node at $[r, s]$.

We now tag the nodes with a 1 (called *live* nodes), or a 0 (called *vacant* nodes) – the value at $[r, s]$ is denoted by $w_{r,s}$ – given by the coefficient of t^r in the formal *generating function*

$$G_s(t) = \sum_0^\infty w_{r,s} t^r. \tag{2.11}$$

The backgrounds we consider will always arise from a *generating fraction* $\varphi_s^*(t)/f^*(t)$, where $\varphi_s^*(t)$ and $f^*(t)$ are polynomials with coefficients in GF(2), with $\deg \varphi_s^*(t) < \deg f^*(t)$. The sequence tags can then be read from $G_s(t)$, obtained from the generating fraction by division.

Conversely, if $G_s(t)$ has period n – we note that *any* polynomial $f(t)$ generates† periodic sequences only – we can write

$$\left.\begin{aligned} G_s(t) &= \left(\sum_0^{n-1} w_{r,s} t^r\right)(1+t^n+t^{2n}\ldots) \\ &= \sigma_s^*(t)/(1+t^n) \\ &= \varphi_s^*(t)/f^*(t) \end{aligned}\right\} \tag{2.12}$$

† We *write* $f^*(t) = c_k + c_{k-1}t + \ldots + c_0 t^k$, $f(t) = c_k t^k + c_{k-1} t^{k-1} + \ldots + c_0$ for reciprocal polynomials, *with coefficients in the same order*, frequently detached so that we have a 'number' $c_k c_{k-1} \ldots c_0$ for either. It is usual to say that $f(t)$ generates the sequences, rather than $f^*(t)$, but we have more use for $f^*(t)$ in this paper and shall regard either as the generator. If we *write* coefficients in reverse order we shall use $f'(t)$ and $f'^*(t)$. Thus $f'(t)$ is another version of $f^*(t)$. Consistency is not often important, and may not always have been achieved.

so that the generating function can be recovered. We call the line $s = 0$ the *ground*, or the *root-* or *base-line*; it contains the line of *roots* or live nodes given by $\varphi_0^*(t)/f^*(t)$, a fraction in its lowest terms (and, as we shall see, normally such that $(1+t) \nmid f^*(t)$). We obtain subsequent lines by use of the rule

$$\left.\begin{aligned} t\varphi_s^*(t) &= (1+t)\,\varphi_{s-1}^*(t) \bmod f^*(t) \\ \text{or} \quad t^s\varphi_s^*(t) &= (1+t)^s\varphi_0^*(t) \bmod f^*(t). \end{aligned}\right\} \tag{2.13}$$

It will readily be verified that, within the sector $r \geqslant 0$, $s \geqslant 0$, we have everywhere

$$\left.\begin{aligned} w_{r,\,s+1} &= w_{r,\,s} + w_{r+1,\,s} \\ \text{or} \quad w_{r,\,s} + w_{r+1,\,s} + w_{r,\,s+1} &= 0. \end{aligned}\right\} \tag{2.14}$$

This is the *fundamental rule* for the formation of *permissible backgrounds* for our purposes.

Since every fraction $\varphi_0^*(t)/f^*(t)$ with $\deg \varphi^* < \deg f^*$ generates a *purely* periodic sequence of minimum period n, say, in the ground row, and consequently in subsequent rows (provided $(1+t) \nmid f^*(t)$), we can extend the background, by use of the n-periodicity, to all negative r. We can also extend similarly to all negative s, for we must also have a row-period, m, since the number of possible sequences of period n is finite. However, periodicity need not necessarily start with a repeat of the root-line; a criterion for pure periodicity is that $((1+t)\,\varphi_0^*(t), f^*(t)) = 1$. When this is so we can say that $f^*(t)$ [or $f(t)$] is the *minimum polynomial* generating this *purely periodic background*, complete for all r, s.

2.2. *Forests, copses, tessellations and nets*

The rule (2.14) means that for a C-*triangle*, i.e. a unit triangle of the background with base vertices $[r, s]$ and $[r+1, s]$, and third vertex $[r, s+1]$, has 0 (mod 2) live nodes as vertices, whereas a B-*triangle*, with the opposite orientation (i.e. third vertex $[r+1, s-1]$) may have any number 0, 1, 2 or 3 of live nodes.

Thus a live node at level $s+1$, at a vertex of a C-triangle has just one corresponding live node on the base of this triangle; we may join these two vertices by a unit line *not* parallel to the ground. If this is done throughout the background, for all r and for all s (or for all $s \geqslant 0$), that is, if all pairs of adjacent live nodes at different s-levels are joined, the result is a *periodic forest of stunted trees* (see Miller 1970). If *all* adjacent pairs of live nodes are connected we get a *tessellation*; in each case the result is doubly periodic with periods n and m.

We need also to study backgrounds defined by a finite sequence of r consecutive nodes along the ground; this will result in a triangle of tagged nodes. If joined as trees it will be called a *copse*, if as a tessellation it will be called a *net*.

The rule (2.14) has triangular symmetry so we could equally well choose the axis of s as ground, or the axis of $q = -r-s$. We can thus obtain *three* distinct forests from a single background, though only *one* tessellation. Likewise a finite background gives three copses, or one net.

The three forests connected with a background may differ in periods n, m, and in generating polynomial $f^*(t)$. A major object of this paper is to characterize and enumerate cases where the three forests are identical, and the tessellation has the rotational symmetry of a triangle.

2.3. *Combination and decomposition of backgrounds; addition and alternation*

The rule (2.14) is such that if we combine two backgrounds, node to node, adding the tags modulo 2, the result is a background for which (2.14) still holds. That is, we can *add* forests or tessellations, and the sum corresponds to the forest or tessellation generated by the sum of the generating fractions for the constituent forests.

We can likewise split generating fractions into partial fractions, with denominators $f_i^*(t)$ that are powers of irreducible polynomials; the background will then also be separable into corresponding constituents.

A direct consequence of (2.14) is the relation

$$w_{r,s} + w_{r+2,s} + w_{r,s+2} = 0.$$

Thus, if we pick alternate nodes from alternate rows, to give a background of equilateral triangles with edge two units, the corresponding live nodes provide a permissible background. This is the process of *alternation*. If the periods m and n are even in the original background, they will be halved by alternation. The process yields four separate forests; when building a forest by the reverse process, any two may be chosen arbitrarily, the other two are then determined. If n is odd, alternation leaves it unchanged and produces four identical forests.

An even period can arise only from a generating polynomial $f^*(t)$ that has a repeated factor, say $[p^*(t)]^\alpha$, where $p^*(t)$ is an irreducible polynomial. Then each forest obtained by alternation may have factor at most $[p^*(t)]^\beta$, where $\beta = [\frac{1}{2}(\alpha+1]$. Thus by alternation, repeated if necessary, we can reduce consideration to denominators without repeated factors.

Thus, by use of partial fractions and alternation we can construct any forest by combining forests generated by single irreducible polynomials only.

2.4. *Periods and cells*

The fraction $1/f^*(t)$ generates a sequence and, if $(1+t, f^*(t)) = 1$, a background with base-period n, where n is the period of $f^*(t)$ such that $f^*(t) \mid (1+t^n), f^*(t) \nmid (1+t^r), r < n$. Then any fraction $\varphi_0^*(t)/f^*(t)$, $\deg \varphi_0^* < \deg f^*$, generates a sequence of period n if $f^*(t)$ is irreducible. If not, we can say only that the sequence has period n', and that $n' | n$.

Any purely periodic background generated by $f^*(t)$ will have also a least row-period m. We can show, then, for any $f^*(t)$, not necessarily irreducible, but with $((1+t), f^*(t)) = 1$, that if $n = 2^j\nu$ with $\nu | 2^e \pm 1$, then $m | 2^j\mu$, where $\mu | 2^e - 1$ (see Miller 1970, §4.5). Note that m and n are always exactly divisible by the same power of 2, here 2^j.

The repetitions of $[0, 0]$ with least r and s are $[n, 0]$ and $[\rho, m]$ for some ρ. These, with the point $[\rho+n, m]$ give a minimum cell for the lattice of repetitions of $[0, 0]$. This contains $2C = 2mn$ unit triangles of the background. If the three forests of the background have periods (n_1, m_1), (n_2, m_2), (n_3, m_3) clearly

$$C = m_1 n_1 = m_2 n_2 = m_3 n_3. \tag{2.41}$$

The triangle $[0, 0], [\rho, m], [n, 0]$ may be isosceles on its base $[0, 0], [n, 0]$. If so, we say that m is an S-period S, implying that the point $[\rho, m]$ is unaltered if we reverse the original sequence, corresponding to generation by the reciprocal polynomal $f'(t)$ or $f'^*(t)$. If these points of repetition at $s = m$ differ, we can find an S-period $S = km$, $k(\frac{1}{2}m+\rho) = \frac{1}{2}n \bmod n$ and $[0, 0]\,[k\rho \bmod n,\ km]$ $[n, 0]$ is isoscleles. The fact that m, n are exactly divisible by the same power 2^j implies that the S-period exists, and that $2S$ is the first period for which nodes match for the same values of x, i.e. for which we have repetition in rectangles. Each forest may have a distinct S-period.

Finally, the isosceles triangle may be equilateral and we have a T-period (only one for each background). If not, we may find a period T (a multiple of S), such that $\kappa m = \lambda n = T$ to give a T-period and an equilateral triangle with base parallel to the ground.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

It is also possible to have a smaller equilateral lattice that is skew, in some cases, giving a skew diamond unit cell.

3. Copses: definition, symmetry, and enumeration

3.1. *Definition*

A *copse* is defined with a finite background of nodes, forming an equilateral triangle, having tags given uniquely by the rule of § 2.14 from an arbitrary set of tags on the finite base line. The *size* of the copse is the number of nodes in the base line.

A *copse proper* is obtained by joining each live node to the unique live node at unit distance from it in the preceding (lower) line. A *net* is obtained by joining also live nodes adjacent in the direction parallel to the base.

If the background is part of an infinite purely periodic background, the copse forms part of the corresponding forest, and the net part of the corresponding tessellation.

We need first to obtain properties of copses. These are relatively simple to determine, and we may construct tessellations from appropriate copses contained in them.

For the consideration of copses and tessellations in general, and of their symmetries, we may use a background tessellation of adjacent regular hexagons, one centred at each node, and coloured yellow if the node is vacant, and blue it if is live. This is used in plates 1 and 2, and it will be noted that the distinction between copse and net, or between forest and tessellation has been lost.

3.2. *Symmetry types of copses*

Consider a copse of size n, with vectors $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})^{\mathrm{T}}$, $\boldsymbol{b}, \boldsymbol{c}$ representing the tags listed counter-clockwise round its outside, $\boldsymbol{a}$ being the base vector. We will use lower case letters for unsymmetric vectors, capital letters for symmetric vectors, and $\boldsymbol{a}$, $\boldsymbol{a}'$ for vectors each of which is the reverse of the other.

A given finite background can stand on a base in six distinct ways, giving 6, 3, 2 or 1 distinct copses for the four possible symmetry arrangements shown in diagram 1.

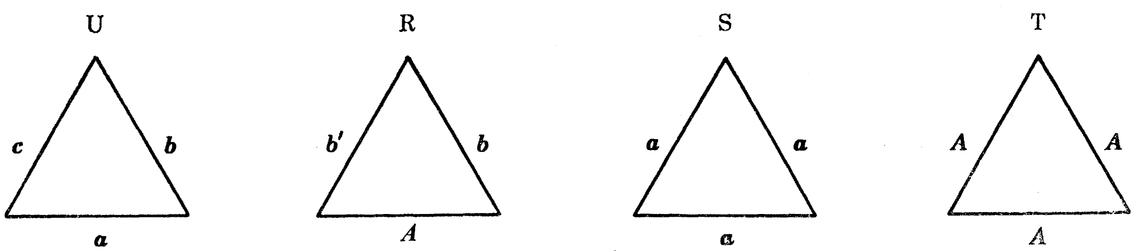


Diagram 1. Symmetry arrangements of copses.

These are respectively unsymmetric U-copses, reflexive R-copses, rotational or skew-symmetric S-copses, and triangular or T-copses. Each, of course, gives just one net.

We now count possible base-vectors, including $(\mathbf{0})$; these are 2^n in all; U-, R-, S-, T-copses represent 6, 3, 2 and 1 vectors respectively.

3.3. *Enumeration of copses and nets*

See plate 1 for illustrations.

Let $u(n)$, $r(n)$, $s(n)$, $t(n)$ represent the number of copses of corresponding type, and $p(n)$ the number of distinct nets. Then clearly

$$6u(n)+3r(n)+2s(n)+t(n) = 2^n. \tag{3.31}$$

Now consider *reflexive* sequences. The number of these depends on the parity of n, and there is one distinct such sequence in each R-background and in each T-background. We thus obtain, with $n = 2m$ or $n = 2m+1$

$$\left.\begin{aligned} r(2m)+t(2m) &= 2^m \\ r(2m+1)+t(2m+1) &= 2^{m+1}. \end{aligned}\right\} \tag{3.32}$$

$$\begin{array}{c} 1 \\ a\ \bar{a} \\ a\ 0\ \bar{a} \\ \bar{a}\ 1\ 1\ a \\ a\ 1\ 0\ 1\ \bar{a} \\ \bar{a}\ 1\ 0\ 0\ 1\ a \\ \bar{a}\ 0\ 1\ 1\ 1\ 0\ a \\ 1\ a\ a\ \bar{a}\ a\ \bar{a}\ \bar{a}\ 1 \end{array}$$

Diagram 2.

Rotational or S-symmetry needs a little more care. If we have an S- or T-background of size n, we can add a row all round, retaining S-symmetry (we may lose reflexion), to give a background of size $n+3$. We may do this in just two ways, as we may see by studying the diagram for $n = 5$. This has a T-background. We can choose a arbitrarily as 0 or 1; the other tags are then determined, with $\bar{a} = a+1$. We thus obtain two possible S-copses or T-copses (S-copses in diagram 2). Initially we have (2, 1, 2) possibilities for $n = (1, 2, 3)$, namely

$$0 \qquad 1 \qquad \begin{array}{c} 0 \\ 0\ 0 \end{array} \qquad \begin{array}{c} 0 \\ 0\ 0 \\ 0\ 0\ 0 \end{array} \qquad \begin{array}{c} 0 \\ 1\ 1 \\ 0\ 1\ 0 \end{array}$$

Finally, noting that each S-background gives, by reflexion, two copses, we have, with $n = 3m+0$, 1, or 2:

$$\left.\begin{aligned} 2s(3m)+t(3m) &= 2^m \\ 2s(3m+1)+t(3m+1) &= 2^{m+1} \\ 2s(3m+2)+t(3m+2) &= 2^m \end{aligned}\right\} \tag{3.33}$$

Triangular or T-symmetry can be retained when adding a row all round in two ways when n is even (the middle element in the new base provides the choice). When n is odd, however, the symmetry can be retained only if the middle element in the bottom row is zero – that is, in just half the cases; the new row can then be added in two ways. Thus when n is odd, we get as many new copses of sizes $n+3$ as we had of size n. We deduce, starting with the possibilities for $n = 1, 2, 3$, that

$$\left.\begin{aligned} t(6m) &= 2^m & \quad t(6m+3) &= 2^{m+1} \\ t(6m+1) &= 2^{m+1} & \quad t(6m+4) &= 2^{m+1} \\ t(6m+2) &= 2^m & \quad t(6m+5) &= 2^{m+1}. \end{aligned}\right\} \tag{3.34}$$

We may now evaluate $r(n)$, $s(n)$, $u(n)$ in succession.

The total number of nets $p(n)$ is

$$p(n) = u(n)+r(n)+s(n)+t(n)$$
$$= \tfrac{1}{6}[\{6u(n)+3r(n)+2s(n)+t(n)\}+3\{r(n)+t(n)\}+2\{2s(n)+t(n)\}]$$

whence (see van Wijngaarden 1966)

$$\left.\begin{aligned} p(6m) &= \tfrac{1}{6}(2^{6m}+3.2^{3m}+2.2^{2m})\\ p(6m+1) &= \tfrac{1}{6}(2^{6m+1}+3.2^{3m+1}+2.2^{2m+1})\\ p(6m+2) &= \tfrac{1}{6}(2^{6m+2}+3.2^{3m+1}+2.2^{2m})\\ p(6m+3) &= \tfrac{1}{6}(2^{6m+3}+3.2^{3m+2}+2.2^{2m+1})\\ p(6m+4) &= \tfrac{1}{6}(2^{6m+4}+3.2^{3m+2}+2.2^{2m+2})\\ p(6m+5) &= \tfrac{1}{6}(2^{6m+5}+3.2^{3m+3}+2.2^{2m+1}). \end{aligned}\right\} \qquad (3.35)$$

See also table 1.

TABLE 1. NUMBERS OF COPSES

n	$t(n)$	$s(n)$	$r(n)$	$u(n)$	$p(n)$
0	1	0	0	0	1
1	2	0	0	0	2
2	1	0	1	0	2
3	2	0	2	0	4
4	2	1	2	1	6
5	2	0	6	2	10
6	2	1	6	7	16
7	4	2	12	14	32
8	2	1	14	35	52
9	4	2	28	70	104
10	4	6	28	154	192
11	4	2	60	310	376
12	4	6	60	650	720
13	8	12	120	1 300	1 440
14	4	6	124	2 666	2 800
15	8	12	248	5 332	5 600
16	8	28	248	10 788	11 072
17	8	12	504	21 588	22 112
18	8	28	504	43 428	43 968
19	16	56	1 008	86 856	87 936
20	8	28	1 016	174 244	175 296
21	16	56	2 032	348 488	350 592
22	16	120	2 032	697 992	700 160
23	16	56	4 080	1 396 040	1 400 192
24	16	120	4 080	2 794 120	2 798 336
25	32	240	8 160	5 588 240	5 596 672
26	16	120	8 176	11 180 680	11 188 992
27	32	240	16 352	22 361 360	22 377 984
28	32	496	16 352	44 730 896	44 747 776
29	32	240	32 736	89 462 032	89 495 040
30	32	496	32 736	178 940 432	178 973 696
31	64	992	65 472	357 880 864	357 947 392
32	32	496	65 504	715 794 960	715 860 992
33	64	992	131 008	1 431 589 920	1 431 721 984
34	64	2 016	131 008	2 863 245 344	2 863 378 432
35	64	992	262 080	5 726 491 680	5 726 754 816
36	64	2 016	262 080	11 453 114 400	11 453 378 560

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

3.4. *Groups of copses and bases for ground-vectors*

Copses may be added, and so those of size n form an additive group of order 2^n. Likewise *reflexive copses*, consisting of R- and T-copses, form a sub-group of order $2^{[\frac{1}{2}(n+1)]}$, in which $[x]$ indicates the integer part of x in the usual way.

There is similarly a group of *rotational copses*, consisting of S- and T-copses, of order 2^m, 2^{m+1}, 2^m when $n = 3m$, $3m+1$, $3m+2$.

Finally there is a group of copses with complete triangular symmetry; the order is 2^m when $n = 6m+i$, $i = 1, 3, 4, 5, 6, 8$.

A basis for the complete group of ground vectors is given by the unit vectors $(1, 0, 0, \ldots, 0)$ $(0, 1, 0, 0, \ldots, 0)$, …, $(0, 0, 0, \ldots, 0, 1)$. Likewise a basis for the ground vectors of the reflexive copses is given by the symmetric vectors with 1 or 2 units; e.g. for $n = 5$ these are $(1, 0, 0, 0, 1)$, $(0, 1, 0, 1, 0)$, $(0, 0, 1, 0, 0)$.

Sets of basic vectors for the S- and T-copses will be developed fully in § 4.

The sets of *nets* consisting of distinct patterns, do not, of course, all form groups.

4. Copses: matrix theory

4.1. *The Pascal matrices. Rotational vectors*

We have seen in Miller 1970 (§3.4) that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ of § 3.2 above satisfy the matrix relations

$$\boldsymbol{c} = \boldsymbol{A}\boldsymbol{a}, \quad \boldsymbol{b} = \boldsymbol{A}\boldsymbol{c}, \quad \boldsymbol{a} = \boldsymbol{A}\boldsymbol{b}, \tag{4.11}$$

in which $\boldsymbol{A}$ is the $\boldsymbol{n} \times \boldsymbol{n}$ matrix.

$$\boldsymbol{A} = \boldsymbol{A}_n = \left(\begin{array}{ccccccc} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 1 & & \\ 1 & 1 & 1 & 1 & & & \\ 1 & 0 & 1 & & & & \\ 1 & 1 & & & & \boldsymbol{0} & \\ 1 & & & & & & \end{array} \right).$$

This we call a ‘Pascal’ matrix of binomial coefficients, here reduced modulo 2. Thus the jth element in the ith row is the parity of the coefficient of t^{j-1} in $(1+t)^{n-i}$.

We deduce that

$$\boldsymbol{A}^3 = \boldsymbol{I}, \tag{4.12}$$

whatever the size of $\boldsymbol{A}$.

Consider now $\boldsymbol{A}_n$. This has at most n independent eigenvectors, row or column. But we already have 2^n column eigenvectors for $\boldsymbol{A}^3$, namely, all the n-vectors with coefficients in GF(2), derivable from a basis of n distinct vectors. Hence all n eigenvalues of $\boldsymbol{A}^3$ are unity, and those of $\boldsymbol{A}$ are all cube roots of unity.

Now, *rotational copses*, i.e. S- and T-copses, arise from what we shall call *rotational* (*column*) *vectors*, which correspond to unit eigen-values of $\boldsymbol{A}$, since $\boldsymbol{a} = \boldsymbol{A}\boldsymbol{a}$; we know there are 2^m, 2^{m+1}, 2^m of these for $n = 3m$, $3m+1$, $3m+2$, and so bases respectively of m, $m+1$ or m rotational vectors. The remaining $2m$, $2m$ or $2m+2$ vectors of the full basis are equally divided between the eigen-values ω and ω^2, and combine symmetrically to give triplets of vectors with all elements in GF(2).

4.2. *Rotational polynomials*

If a $(k+1)$-vector of a forest is generated by

$$f_1^*(t) = \alpha_k + \alpha_{k-1} t + \alpha_{k-2} t^2 + \ldots + \alpha_2 t^{k-2} + \alpha_1 t^{k-1} + \alpha_0 t^k$$

of degree k, and we write $\boldsymbol{f}_1 = (\alpha_0, \alpha_1, \ldots, \alpha_k)^{\mathrm{T}}$, the vector of coefficients in the *reverse order*,† then

$$\boldsymbol{f}_1^{\mathrm{T}} \boldsymbol{a} = 0. \tag{4.21}$$

We know also that, for vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, reading counter-clockwise round a copse,

$$\boldsymbol{c} = \boldsymbol{A}\boldsymbol{a}, \quad \boldsymbol{b} = \boldsymbol{A}^2\boldsymbol{a}. \tag{4.22}$$

If, then, we have several $(k+1)$-copses from the same background generated by $f_1^*(t)$, with enough independent vectors $\boldsymbol{a}$ so that they determine $\boldsymbol{f}_1$ uniquely, and if we form a matrix $\boldsymbol{S}_1$ with the $\boldsymbol{a}$ as its columns, we know that

$$0 = \boldsymbol{f}_1^{\mathrm{T}} \boldsymbol{S}_1 = \boldsymbol{f}_1^{\mathrm{T}} \boldsymbol{A} . \boldsymbol{A}^2 \boldsymbol{S}_1 = \boldsymbol{f}_1^{\mathrm{T}} \boldsymbol{A}^2 . \boldsymbol{A} \boldsymbol{S}_1. \tag{4.23}$$

Then $\boldsymbol{S}_2 = \boldsymbol{A}^2 \boldsymbol{S}_1$ is a set of vectors $\boldsymbol{b}$ from a forest generated by $f_2^*(t)$, and $\boldsymbol{S}_3 = \boldsymbol{A}\boldsymbol{S}_1$ is a set of vectors $\boldsymbol{c}$ from a forest generated by $f_3^*(t)$, so that

$$\boldsymbol{f}_1^{\mathrm{T}} \boldsymbol{A} = \boldsymbol{f}_2^{\mathrm{T}}, \quad \boldsymbol{f}_1^{\mathrm{T}} \boldsymbol{A}^2 = \boldsymbol{f}_3^{\mathrm{T}}. \tag{4.24}$$

We may then readily verify (Miller 1970, §3.5) that

$$f_2(t) = f_1^*(t+1), \quad f_3(t) = f_2^*(t+1), \quad f_1(t) = f_3^*(t+1). \tag{4.25}$$

This indicates that $f_1(t), f_2(t), f_3(t)$ have similar factorization characteristics. All are irreducible if any one is; all factorize into the same number of factors with the same set of degrees, and these factors may be arranged into corresponding triplets of identical or related polynomials of the same degree. This statement applies to polynomials with $\alpha_0 = \alpha_k = 1$. It can be extended to allow either or both to be zero, i.e. to general $(k+1)$-vectors in GF(2), provided that we consider the polynomials t, $t+1$, 1 or 10, 11, 01 as a triplet of polynomials of 'degree' 1; they form a triplet for $\boldsymbol{A}_2$. In this case it is the length of the vector, rather than the actual degree of the polynomial, that is relevant; the total 'degree' is always k for $(k+1)$-vectors, but a power of 01 may be involved indicating the number of initial zeros, or of 10 giving the number of final zeros.

We can now examine the structure of the row eigensystem of $\boldsymbol{A}$. The eigenvalues of $\boldsymbol{A}^3$ are all unity, and a complete group of 2^n eigenvalues exists. The eigenvalues of $\boldsymbol{A}$ are 1 and the two cube-roots of unity. Corresponding to eigenvalues 1 of $\boldsymbol{A}$ we have *rotational* (*row*)-*eigenvectors*, which we shall call *rotational polynomials*, since all the polynomial vectors $\boldsymbol{f}_i$ such that $f_i^*(t)$ generates rotational tessellations turn up here, and comprise the set of all such vectors having $\alpha_0 = \alpha_k = 1$.

Corresponding to the eigenvalues ω, ω^2 we have eigenvectors whose elements are not in GF(2), but which can be combined to give (ordered) triplets of vectors that have elements in GF(2), namely triplets $(\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$ or $(\boldsymbol{f}_1', \boldsymbol{f}_2', \boldsymbol{f}_3')$. These in turn give triplets of generating polynomials

† The reversal is needed for convenient matrix multiplication. Most of §§4 and 5 is concerned with eigenvectors of the Pascal matrix $\boldsymbol{A}$, in which vectors like $\boldsymbol{f}_1$ and $\boldsymbol{a}$ play a major part. We shall therefore write vectors of detached coefficients in the order in which they occur in these vectors in these sections, and shall abandon a fixed order for writing terms in polynomials $f(t)$, $f^*(t)$ so that we can avoid reversals when comparing with $\boldsymbol{f}$. This suggests use of $f^*(t)$ with high powers written first.

$(f_1^*(t), f_2^*(t), f_3^*(t))$ or $(f_1'^*(t), f_3'^*(t), f_2'^*(t))$ applying to a single background, or *rotational triplets of polynomials*. The triplets are cyclic, i.e. the triplet $(\boldsymbol{f}_2, \boldsymbol{f}_3, \boldsymbol{f}_1)$ may be combined (by multiplication) in use with the triplet $(\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$ so far as forests are concerned to give the triplet

$$(f_1^*(t) f_2^*(t), f_2^*(t) f_3^*(t), f_3^*(t) f_1^*(t)),$$

and even for tessellations, the backgrounds corresponding to different orientations may be added.

The multiplicative properties mentioned indicate that rotational polynomials form a multiplicative ring, in fact, an integral domain, also that when $f_1^*(t), f_2^*(t), f_3^*(t)$ differ they are not individually rotational, nevertheless their product $f_1^*(t) f_2^*(t) f_3^*(t)$ *is* rotational.

4.3. *Row eigen-vectors of* $\boldsymbol{A}$. *A basis for rotational polynomials*

We are now in a position to obtain a basis for rotational row-vectors and polynomials connected with $\boldsymbol{A}_n$ for any degree $k+1 = n$.

Firstly, we note that for $n = 3\lambda+1$, $3\lambda+3$, $3\lambda+5$ there are 2^λ rotational polynomials having $\alpha_0 = \alpha_k = 1$ and, in all, $2^{\lambda+1}-1$ vectors, including those with zeros at both ends. Now $b = t^2+t+1$ is a rotational polynomial (we shall see later it is the only irreducible such polynomial that is reflexive, and the only irreducible and rotational one whose degree is not a multiple of 3). Hence the vectors for $n = 3\lambda+3$ and for $n = 3\lambda+5$ are exactly those for $n = 3\lambda+1$ multiplied by b and b^2 respectively. We need therefore consider only $n = 3\lambda+1$ any further.

Next we note that the only irreducible polynomial with constant term zero is t. This is a member of the triplet of 'first' degree polynomials $(t, t+1, 1)$ associated with $\boldsymbol{A}_2$. Any triplet for which we do not have $\alpha_0 = \alpha_k = 1$ for every member must have this triplet as a 'factor'. This triplet yields the unique rotational polynomial 0 1 1 0 or $0.t^3+t^2+t+0 = a$, associated with $\boldsymbol{A}_4$ (not $\boldsymbol{A}_3$), that has not got $\alpha_0 = \alpha_k = 1$. Any rotational vector with zeros at its ends, always the same number, i, say, at both ends, will have a^i as a factor.

The full set of rotational polynomials of $n = 3\lambda+1$ then has 2^λ rotational polynomials with $\alpha_0 = \alpha_k = 1$, also $2^{\lambda-1}$ with a zero at each end, $2^{\lambda-2}$ with 2 zeros at each end, and so on, those polynomials with zeros being the polynomials for $n = 3\lambda-2$ each multiplied by a. Thus as a basis for the vectors we see that we need one vector with each possible number of zeros at the two ends, commencing with a power of a. It is easily verified that the following set will suffice for $n = 3\lambda+1$: a^λ, $ca^{\lambda-1}$, $b^3a^{\lambda-2}$, $b^3ca^{\lambda-3}$, ... (in these $c = f^*(t) = t^3+t+1$, $c = 1011$) finishing with a polynomial without a factor a. There are just $\lambda+1$ polynomials in the set, giving $2^{\lambda+1}-1$ vectors in all. Other sets are possible, according to need. For example, when using polynomials of the full degree, we merely have to assure the presence of the last member of the basis in every combination. In §4.8 we shall use a set chosen in order to pick out irreducible polynomials; this set needs as great a spread of small factors as possible, and the set used is

$$a^\lambda,\ a^{\lambda-1}c,\ a^{\lambda-2}cc',\ a^{\lambda-3}c^2c',\ a^{\lambda-4}b^3cc',\ a^{\lambda-5}b^3c^2c',\ a^{\lambda-6}b^3c^2c'^2,\ a^{\lambda-7}b^3cc'p_9 \text{ etc.}$$

4.4. *A basis for rotational (column-)vectors*

The rotational column-vectors do not have the full multiplicative structure of the row vectors. On the other hand, a connection between sizes of copses n and $n-3$ obtained by stripping off a row all round is a satisfactory substitute. We can obtain a partial multiplicative structure by demonstrating that any rotational polynomial-vector may be used to give a copse.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Consider the tree generated by an isolated root. This expands indefinitely to give an unbroken row of 2^j units in row 2^j-1 and two well separated copies of the single root at row 2^j. If we start with a lone polynomial vector, $\boldsymbol{f}$, on the ground we shall likewise, in row 2^j, have two separate copies of $\boldsymbol{f}$ provided $2^j \geqslant \deg f_1$.† We consider one of the two 'anti-copses' that end with these copies of f, together with one zero at the end nearer the other copy. In this the top line on the right gives the polynomial $tf_1(t)$), and the outer edge consists of units throughout. The inner edge has a polynomial $tg^*(t)$, say, running downward and outward from the top left corner of the 'anti-copses' (see diagram 3).

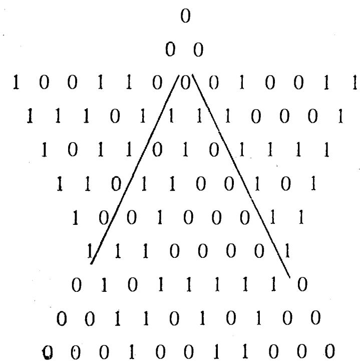


DIAGRAM 3.

Write

$$\left.\begin{aligned} tf_1(t) &= \alpha_0 t + \alpha_1 t^2 + \alpha_2 t^3 + \dots \\ tg^*(t) &= \gamma_0 t + \gamma_1 t^2 + \gamma_2 t^3 + \dots \end{aligned}\right\} \qquad (4.41)$$

Then $tf_1(t)$ is generated entirely by sub-trees from the polynomial $tg^*(t)$, omitting the initial left hand branch of the sub-tree from each node γ_i (this merely yields a contribution to γ_{i-1}, already known and allowed for). Thus

$$\left.\begin{aligned} tf_1(t) &= t\gamma_0 + t\gamma_1(t+1) + t\gamma_2(t+1)^2 + \dots \\ &= tg^*(t+1) \end{aligned}\right\} \qquad (4.42)$$

so that $g^*(t) = f_3(t)$ (see Miller 1970, §3.5).

Similarly we have $f_2(t)$ up the inner edge of the 'anti-copse' on the left. Thus the normal copse obtained by filling in the zeros outside the initial tree is a copse with $\boldsymbol{f}_1, \boldsymbol{f}_3, \boldsymbol{f}_2$ in the centres of the vectors. If $\deg f_1 = k$, then the size of the copse is $2.2^j - k - 1$ for some $2^j \geqslant k$.

Copses of this type suffice to give a basis for rotational vectors in all cases, using only 2^j and 2^{j+1} where $2^j \geqslant k \geqslant 2^{j-1}$. Each copse gives a basis for size

$$2.2^j - k - 1 - 3r, \quad r = 0, 1, 2, \dots$$

by stripping round the edges r times, where $f(t)$ is now a rotational polynomial. The corresponding vectors are given by

$$f(t) \times 1^{2^j-k-1-2r}(t+1)^r t^{2^j-k-1-2r}$$

so long as zeros only are stripped from the ends, i.e. so long as

$$4r \leqslant 2(2^j - k - 1).$$

† Adjacent copies if $\deg f_1 = 2^j - 1$; separated by one or more zeros if $\deg f_1 < 2^j - 1$. The rest of the argument needs slight modification in the former case, since there is no intervening zero between copies, but it still holds.

Table 2. Table of bases for rotational vectors

$$\left.\begin{array}{l} 1^{2^j-2i-1}(t+1)^i\, t^{2^j-2i-1} \text{ of length } n = 2\cdot 2^j-3i-1 \\ 1^{2^j-2\lambda-2i-1}b^\lambda(t+1)^i\, t^{2^j-2\lambda-2i-1} \text{ of length } n = 2\cdot 2^j-2\lambda-3i-1 \\ 1^{2^j-2\lambda-2i-4}b^\lambda c(t+1)^i\, t^{2^j-2\lambda-2i-4} \text{ of length } n = 2\cdot 2^j-2\lambda-3i-4 \end{array}\right\} \quad \begin{array}{l} b = t^2+t+1, \quad c = t^3+t+1. \\ 0 \leqslant 4i \leqslant 2(2^j-k-1) \end{array}$$

i \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
0	1	.	1	c	b	.	1	b^2c	b^3	bc	b^2	c	b	.	1	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b	.	1	$b^{14}c$	b^{15}	$b^{13}c$	b^{14}	$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c
1				1			bc	b^2	c	b		1			b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1			$b^{13}c$	b^{14}	$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5
2						c	b		1					b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1					$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c
3						1							b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1							$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2
4												b^2c	b^3	bc	b^2	c	b		1									$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b	
5											bc	b^2	c	b		1											b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1		
6										c	b		1													b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1					
7										1															b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1								
8																								b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1											
9																							b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1														
10																						b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1																	
11																					b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1																				
12																				b^2c	b^3	bc	b^2	c	b		1																							
13																			bc	b^2	c	b		1																										
14																		c	b		1																													$b^{16}c$
15																		1																															$b^{15}c$	b^{16}
16																																																$b^{14}c$	b^{15}	$b^{13}c$
17																																															$b^{13}c$	b^{14}	$b^{12}c$	b^{13}
18																																														$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$
19																																													$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}
20																																												$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c
21																																											b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7
22																																										b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c
23																																									b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4
24																																								b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc
25																																							b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b
26																																						b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1	
27																																					b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1				
28																																				b^2c	b^3	bc	b^2	c	b		1							
29																																			bc	b^2	c	b		1										
30																																		c	b		1													
31																																		1																

(*Facing p.* 610)

Table 2. Table of bases for rotational vectors

$$\left.\begin{array}{ll} 1^{2^j-2i-1}(t+1)^i\, t^{2^j-2i-1} & \text{of length } n = 2\cdot 2^j - 3i - 1 \\ 1^{2^j-2\lambda-2i-1} b^\lambda (t+1)^i\, t^{2^j-2\lambda-2i-1} & \text{of length } n = 2\cdot 2^j - 2\lambda - 3i - 1 \\ 1^{2^j-2\lambda-2i-4} b^\lambda c\,(t+1)^i\, t^{2^j-2\lambda-2i-4} & \text{of length } n = 2\cdot 2^j - 2\lambda - 3i - 4 \end{array}\right\} \quad \begin{array}{l} b = t^2+t+1, \quad c = t^3+t+1. \\ 0 \leqslant 4i \leqslant 2(2^j-k-1) \end{array}$$

i \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	.	1	c	b	.	1	b^2c	b^3	bc	b^2	c	b	.	1	b^6c	b^7	b^5c
1				1			bc	b^2	c	b		1			b^5c	b^6	b^4c	b^5
2						c	b		1					b^4c	b^5	b^3c	b^4	b^2c
3						1							b^3c	b^4	b^2c	b^3	bc	b^2
4												b^2c	b^3	bc	b^2	c	b	
5											bc	b^2	c	b		1		
6										c	b		1					
7										1								
8																		
9																		
10																		
11																		
12																		
13																		
14																		c
15																		1
16																		
17																		
18																		
19																		
20																		
21																		
22																		
23																		
24																		
25																		
26																		
27																		
28																		
29																		
30																		
31																		

i \ n	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
0	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b	.	1	$b^{14}c$	b^{15}	$b^{13}c$	b^{14}
1	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1			$b^{13}c$	b^{14}	$b^{12}c$	b^{13}	$b^{11}c$
2	b^3	bc	b^2	c	b		1					$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$	b^{11}
3	c	b		1							$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c
4	1									$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8
5									b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c
6								b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5
7							b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c
8						b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2
9					b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b	
10				b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1		
11			b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1					
12		b^2c	b^3	bc	b^2	c	b		1								
13	bc	b^2	c	b		1											
14	b		1														
15																	
16																	
17																	
18																	
19																	
20																	
21																	
22																	
23																	
24																	
25																	
26																	
27																	
28																	
29																	bc
30																c	b
31																1	

(*Facing p.* 610)

$+1, \quad c = t^3+t+1.$

$2(2^j-k-1)$

	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
	$b^{14}c$	b^{15}	$b^{13}c$	b^{14}	$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	$b^{9}c$	b^{10}	$b^{8}c$	b^{9}	$b^{7}c$	b^{8}	$b^{6}c$	b^{7}	$b^{5}c$
^{3}c	b^{14}	$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	$b^{9}c$	b^{10}	$b^{8}c$	b^{9}	$b^{7}c$	b^{8}	$b^{6}c$	b^{7}	$b^{5}c$	b^{6}	$b^{4}c$	b^{5}
3	$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	$b^{9}c$	b^{10}	$b^{8}c$	b^{9}	$b^{7}c$	b^{8}	$b^{6}c$	b^{7}	$b^{5}c$	b^{6}	$b^{4}c$	b^{5}	$b^{3}c$	b^{4}	$b^{2}c$
^{0}c	b^{11}	$b^{9}c$	b^{10}	$b^{8}c$	b^{9}	$b^{7}c$	b^{8}	$b^{6}c$	b^{7}	$b^{5}c$	b^{6}	$b^{4}c$	b^{5}	$b^{3}c$	b^{4}	$b^{2}c$	b^{3}	bc	b^{2}
0	$b^{8}c$	b^{9}	$b^{7}c$	b^{8}	$b^{6}c$	b^{7}	$b^{5}c$	b^{6}	$b^{4}c$	b^{5}	$b^{3}c$	b^{4}	$b^{2}c$	b^{3}	bc	b^{2}	c	b	
c	b^{8}	$b^{6}c$	b^{7}	$b^{5}c$	b^{6}	$b^{4}c$	b^{5}	$b^{3}c$	b^{4}	$b^{2}c$	b^{3}	bc	b^{2}	c	b		1		
	$b^{5}c$	b^{6}	$b^{4}c$	b^{5}	$b^{3}c$	b^{4}	$b^{2}c$	b^{3}	bc	b^{2}	c	b		1					
c	b^{5}	$b^{3}c$	b^{4}	$b^{2}c$	b^{3}	bc	b^{2}	c	b		1								
	$b^{2}c$	b^{3}	bc	b^{2}	c	b		1											
	b^{2}	c	b		1														
		1																	

																$b^{16}c$
															$b^{15}c$	b^{16}
														$b^{14}c$	b^{15}	$b^{13}c$
													$b^{13}c$	b^{14}	$b^{12}c$	b^{13}
												$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$
											$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	$b^{9}c$	b^{10}
										$b^{10}c$	b^{11}	$b^{9}c$	b^{10}	$b^{8}c$	b^{9}	$b^{7}c$
									$b^{9}c$	b^{10}	$b^{8}c$	b^{9}	$b^{7}c$	b^{8}	$b^{6}c$	b^{7}
								$b^{8}c$	b^{9}	$b^{7}c$	b^{8}	$b^{6}c$	b^{7}	$b^{5}c$	b^{6}	$b^{4}c$
							$b^{7}c$	b^{8}	$b^{6}c$	b^{7}	$b^{5}c$	b^{6}	$b^{4}c$	b^{5}	$b^{3}c$	b^{4}
						$b^{6}c$	b^{7}	$b^{5}c$	b^{6}	$b^{4}c$	b^{5}	$b^{3}c$	b^{4}	$b^{2}c$	b^{3}	bc
					$b^{5}c$	b^{6}	$b^{4}c$	b^{5}	$b^{3}c$	b^{4}	$b^{2}c$	b^{3}	bc	b^{2}	c	b
				$b^{4}c$	b^{5}	$b^{3}c$	b^{4}	$b^{2}c$	b^{3}	bc	b^{2}	c	b		1	
			$b^{3}c$	b^{4}	$b^{2}c$	b^{3}	bc	b^{2}	c	b		1				
		$b^{2}c$	b^{3}	bc	b^{2}	c	b		1							
	bc	b^{2}	c	b		1										
c	b		1													
1																

After that, with 1's lost from the ends when stripping, the vectors are still rotational, but are not related to rotational polynomials $f^*(t)$ in the manner described.

The rotational polynomials $b = t^2+t+1$ and $c = t^3+t+1$ suffice to give a complete basis for rotational vectors as indicated in table 2 (and easily proved by using and extending the table and counting). The basic vectors are

$$1^{2j-2i-1}(t+1)^i\,t^{2j-2i-1} \quad \text{of length} \quad n = 2.2^j-3i-1$$

$$1^{2j-2\lambda-2i-1}b^\lambda(t+1)^i\,t^{2j-2\lambda-2i-1} \quad \text{of length} \quad n = 2.2^j-2\lambda-3i-1$$

$$1^{2j-2\lambda-2i-4}b^\lambda c(t+1)^i\,t^{2j-2\lambda-2i-4} \quad \text{of length} \quad n = 2.2^j-2\lambda-3i-4$$

for $i = 0, 1, 2, \ldots$ so long as $4i \leqslant 2(2^j-k-1)$, $k = 0$, 2λ or $2\lambda+3$.

For example the basis for $n = 12$ is:

$$\text{for } 2^j = 8 \quad \left.\begin{array}{ll} 1^4ct^4 & \text{or} \quad 0\;0\;0\;0\;1\;0\;1\;1\;0\;0\;0\;0 \\ 1^5(t+1)\,t^5 & \text{or} \quad 0\;0\;0\;0\;0\;1\;1\;0\;0\;0\;0\;0 \end{array}\right\}$$

$$\text{for } 2^j = 16 \quad \left.\begin{array}{ll} (t+1)^4b^2c & \text{or} \quad 1\;0\;0\;1\;1\;1\;1\;0\;0\;1\;1\;1 \\ 1(t+1)^5b^2t & \text{or} \quad 0\;1\;1\;1\;1\;0\;0\;1\;1\;1\;1\;0 \end{array}\right\}.$$

Other vectors, keeping each within its group for fixed j are:

$$\text{for } 2^j = 8 \quad 1^4c't^4 = 0\;0\;0\;0\;1\;1\;0\;1\;0\;0\;0\;0$$

$$\text{for } 2^j = 16 \quad (t+1)^4b^2c' = 1\;1\;1\;0\;0\;1\;1\;1\;1\;0\;0\;1.$$

These are both derived from rotational polynomials.

The other rotational row vectors, with contribution from both j-groups, are

0 1 1 1 1 1 1 1 1 1 1 0	$01 \times 11 \times 1111^2 \times 10$
0 1 1 1 0 0 1 0 1 1 1 0	$01 \times 10011 \times 111101 \times 10$
0 1 1 1 0 1 0 0 1 1 1 0	$01 \times 11001 \times 101111 \times 10$
1 1 1 0 0 0 0 1 1 0 0 1	11×10111110111
1 0 0 1 1 0 0 0 0 1 1 1	11×11101111101
1 1 1 0 1 1 0 0 1 0 0 1	p
1 0 0 1 0 0 1 1 0 1 1 1	p'
1 1 1 0 1 0 1 0 1 0 0 1	1101×11111^2
1 0 0 1 0 1 0 1 0 1 1 1	1011×11111^2

Although two of these have a rotational polynomial factor, not one is free of 'general' factors; even so, the resulting copse is, in every case, rotational.

It is worth noting that, if $n = 2^j$, the row and column eigenvector systems are identical, also that for $n = 2^j+i$, $i = -2, 0, 1$, all column vectors belong to one system only. This is not so for $n = 2^j-1$, since the 1 at the top of a column in table 2 belongs to the group on its left.

The sub-groups of vectors for a given value of 2^j are multiples of a group of rotational row-vectors, for the method of this §4.4 works identically for every member of such a group, whether $\alpha_0 = \alpha_k = 1$, or α_0 and/or $\alpha_k = 0$; we have to count end-zeros a little differently in the latter case, distinguishing those that 'belong' to the polynomial from those that do not. For $n = 2^j+i$, the row-polynomials are multiplied by 1^it^i, for $n = 2^j-i$, they are multiplied by $(t+1)^i$

Another simple way to build up a vector basis for rotational n-vectors is simply to use a basis for $(n-3)$-vectors. By adding a row round the copse given by the $(n-3)$-vector, as explained in §3.3, we can obtain a set of independent n-vectors, one from each $(n-3)$-vector. The second n-vector obtainable from each of these copses is found by adding the copse based on the n-vector $(0, 1, 1, \ldots, 1, 0)$; this is independent of the others and is itself obtainable by bordering the zero copse. This last vector is the extra one needed for the n-basis. This method has the disadvantage that we have to know the $(n-3)$-basis first.

5. Enumeration of primary and irreducible polynomials

5.1. *The isomorphism between four multiplicative rings of polynomials*

We first exhibit an isomorphism between four distinct multiplicative rings of polynomials. These are:

(*a*) the ring of all binary polynomials,

(*b*) the ring of symmetric or reflexive polynomials,

(*c*) the ring of rotational polynomials,

(*d*) the ring of triangular, i.e. reflexive and rotational polynomials.

We note that there are 2^k polynomials in the sets for respective degrees k, $2k$, $3k$ and $6k$, and we are proposing to enumerate the *primary polynomials* in each case for degrees k, $2k$, $3k$ or $6k$. These are the polynomials that appear for the first time at that degree, having no factor within the ring.

Consider each ring in turn; in each case we have the unit polynomial 1 for $k = 0$.

(*a*) We consider all polynomials of exact degree k.

For $k = 1$ we have two primary ones, $t+1$ and t, or 11 and 10. This determines the structure, for we know there are 2^k polynomials of exact degree k. So for $k = 2$, we have 3 products $(t+1)^2$, $(t+1)\,t$, and t^2 and one primary t^2+t+1. In this ring the primary polynomials are the *irreducibles*.

We can extend the ring to include multiples of $0.t+1$ (the vector of coefficients 01), i.e. to include polynomials of degree $\leqslant k$. This gives $2^{k+1}-1$ in all.

(*b*) Reflexive polynomials of exact degree $2k$. These also number 2^k, and have $\alpha_0 = \alpha_{2k} = 1$.

For $k = 1$, $n = 2$ we have two primary polynomials, t^2+t+1 and $(t+1)^2$. Again the structure is determinate and identical with that in (*a*) above. For $k = 2$ we have 3 products $(t^2+t+1)^2$, $(t^2+t+1)\ (t+1)^2$, and $(t+1)^4$, and one primary $t^4+t^3+t^2+t+1$.

We can also extend the ring to include reflexive polynomials with zeros at both ends by using multiples of $0.t^2+t+0$, just as we used $0.t+1$ in (*a*), to give $2^{k+1}-1$ members. We also allow for degree $2k+1$, with polynomials equi-numerous with those for degree $2k$, by multiplying every polynomial for degree $2k$ by $t+1$.

(*c*) The third ring is that of rotational polynomials. Here for $n = 3$ we again have two primary polynomials $t^3+t+1 = c$ and $t^3+t^2+1 = c'$. For $n = 6$ we have three products c^2, cc', and c'^2 and one polynomial $b^3 = (t^2+t+1)^3$ appearing for the first time with degree $3k$, and so primary. The structure is thus as before. Yet again we can extend the ring to polynomials with zero coefficients at beginning and end by using the degree 3 rotational multiplier $a = 0.t^3+t^2+t+0$. We can also allow for polynomials of degrees $3k+2$ or $3k+4$, each equi-numerous with those of degree $3k$, by multiplying each polynomial for degree $3k$ by b or b^2 respectively.

(*d*) Finally we have triangular polynomials, both rotational and reflexive. Here for degree $k = 6$ we again have two primary polynomials, $b^3 = 1\ 1\ 0\ 1\ 0\ 1\ 1$ and $cc' = 1\ 1\ 1\ 1\ 1\ 1\ 1$;

for $k = 12$ we have products b^6, b^3cc', $(cc')^2$ and a primary $c_{12} = 10011 \times 11001 \times 11111$ (using detached coefficients). We include polynomials with zero coefficients at the ends by using the multiplier $a^2 = 0010100$. The other 'degrees' with polynomials equi-numerous with those for $6k$ are $6k+2$, 3, 4, 5, and 7; the corresponding multipliers are here b, a, b^2, ab, and ab^2, but *all* with odd degree have end zeros.

Thus all these sets of primary polynomials are counted by Selmer's function $I_2(k)$ (see Selmer 1966, pp. 12, 201).

Now, every polynomial of degree d dividing k is a factor of the *characteristic product* $t^{2^k} + t = 0$, and each zero of each polynomial is represented just once. Thus

$$\sum_{d|k} dI_2(d) = 2^k \tag{5.11}$$

whence by Möbius inversion

$$I_2(k) = \frac{1}{k} \sum_{d|k} \mu(d)\, 2^{k/d}. \tag{5.12}$$

Table 3 lists these numbers to $k = 30$. We also give the number $P_2(k)$ of primitive polynomials, i.e. those for which the period of the sqeuence generated by $f(t)$ is exactly $2^k - 1$; we have

$$P_2(k) = \phi(2^k - 1)/k \tag{5.13}$$

where $\phi(s)$ is Euler's function.

Table 3. Numbers of irreducible, primitive, reflexive and rotational polynomials

k	$I_2(k)$	$P_2(k)$	$R(2k)$	$\frac{1}{2}S(3k)$
1	2	1	1	1
2	1	1	1	0
3	2	2	1	1
4	3	2	2	1
5	6	6	3	2
6	9	6	5	3
7	18	18	9	6
8	30	16	16	10
9	56	48	28	19
10	99	60	51	33
11	186	176	93	62
12	335	144	170	112
13	630	630	315	210
14	1 161	756	585	387
15	2 182	1 800	1 091	728
16	4 080	2 048	2 048	1 360
17	7 710	7 710	3855	2 570
18	14 532	7 776	7 280	4 845
19	27 594	27 594	13 797	9 198
20	52 377	24 000	26 214	17 459
21	99 858	84 672	49 929	33 288
22	190 557	120 032	95 325	63 519
23	364 722	356 960	182 361	121 574
24	698 870	276 480	349 520	232 960
25	1 342 176	1 296 000	671 088	447 392
26	2 580 795	1 719 900	1 290 555	860 265
27	4 971 008	4 202 496	2 485 504	1 657 009
28	9 586 395	4 741 632	4 793 490	3 195 465
29	18 512 790	18 407 808	9 256 395	6 170 930
30	35 790 267	17 820 000	17 895 679	11 930 100

The number $I_2(k)$, then, enumerates the primary polynomials in all four cases. These polynomials are not all irreducible and we need further consideration to sort out those that are.

5.2. *The characteristic products for reflexive, rotational and triangular polynomials*

We have seen that the irreducible polynomials of degree d dividing k are precisely those which form the product $t^{2^k}+t$.

We shall now obtain similar characteristic products for the reflexive and rotational polynomials, and show that there is just one irreducible triangular polynomial, namely t^2+t+1.

Reflexive or symmetric polynomials, if irreducible, must have an odd number of 1's. Thus the degree must be even, $2k$, to have a 1 in the middle. If, now, α is a zero of $f^*(t)$, α^{-1} must also be zero. If $\alpha, \alpha^2, \alpha^{2^2}, \ldots, \alpha^{2^{2k-1}}$ are the zeros, with $\alpha^{2^{2k}}=\alpha$, mod f^*, whence the middle one must be α^{-1} and $\alpha^{2^k}=\alpha^{-1} \bmod f^*$, whence

$$f^*(\alpha)\,|\,\alpha^{2^k+1}+1. \tag{5.21}$$

Hence all reflexive polynomials $f^*(t)$ of degree $2k$ are factors of $t^{2^k+1}+1$.

The whole argument is reversible, so that all factors of this, including those that have degrees dividing k, are symmetric.

For rotational polynomials we consider

$$f^*(t)=t^{\lambda+1}+t+1 \quad \text{and} \quad f'^*(t)=t^{\lambda+1}+t^{\lambda}+1 \quad \text{with} \quad \lambda=2^k.$$

Both of these divide $t^{\lambda^2+\lambda+1}+1$, which in turn divides $t^{\lambda^3-1}+1$, while $f^*(t), f'^*(t)$ divide respectively $t^{\lambda^2+1}+t^{\lambda^2}+1$ and $t^{\lambda^2+1}+t+1$. All three of $t^{\lambda+1}+t+1$, $t^{\lambda+1}+t^{\lambda}+1$ and $t^{\lambda}+t$ are coprime in pairs except for a simultaneous factor t^2+t+1 when k is even.

Consider, for example, $f^*(t)=t^{2^k+1}+t+1$.

Then $$t^{2^k(2^k+1)}=t^{2^k}+1=t^{-1} \bmod f^*$$

or $$t^{2^{2k}+2^k+1}+1=0 \bmod f^*.$$

Also $$t^{2^{2k}+2^k+1}+1\,|\,t^{2^{3k}-1}+1, \quad \text{since} \quad 2^{2k}+2^k+1\,|\,2^{3k}-1.$$

Finally $$0=1+t^{2^{2k}+2^k+1}=1+t^{2^{2k}}(1+t) \bmod f^*$$

so that $$f^*\,|\,1+t^{2^{2k}}+t^{2^{2k}+1}. \tag{5.22}$$

The rest follows by taking reciprocal equations.

Now $t^{2^k+1}+t+1$ and $t^{2^k+1}+t^{2^k}+1$ are both rotational, since replacing t by $t+1$ converts each to the other, its reciprocal. Hence all their factors, which must be of degree dividing $3k$, are irreducible rotational polynomials – irreducible because any rotational triplet would have factors of degree dividing k and so dividing $t^{2^k}+t$, which is prime to $t^{2^k+1}+t+1$ and $t^{2^k+1}+t^{2^k}+1$ (except for the factor t^2+t+1 when k is even – a factor which is special and outside the general scheme).

Note that *all* the composite primary rotational polynomials are factors of $t^{2^k}+t$, and that the triplets of factors of exact degree k give all these composites.

Next we can show that any irreducible rotational polynomial of degree $3k$ must divide one or other of the two polynomials $t^{2^k+1}+t^{2^k}+1$ or $t^{2^k+1}+t+1$. For if $f_1(t)$ is such a polynomial, with zeros $\alpha, \alpha^2, \alpha^{2^2}, \ldots, \alpha^{2^{3k-1}}$, then $f_1^*(t)$ has the zero α^{-1} corresponding to α, and $f_2(t)=f_1^*(t+1)$ has a zero $1+\alpha^{-1}$, so that either $\alpha^{-1}+1=\alpha^{2^j}$ or $(\alpha^{-1}+1)^{2^{j'}}=\alpha$ for some j, j'. Thus either $1+\alpha+\alpha^{2^j+1}=0$ or $1+\alpha^{2^{j'}}+\alpha^{2^{j'}+1}=0$. These divide respectively $\alpha^{2^{3j}}+\alpha$ or $\alpha^{2^{3j'}}+\alpha$ and, since both f_1 and f_2 are of degree $3k$, we must have $j = k$ or $2k$ and, by the relations proved above, $j' = 2k$ or k correspondingly. (The least j or j' might be factors of k and $2k$, by odd divisors, but it is still true as stated.) In either case $f_1(t), f_2(t)$ divide our polynomials, one each.

Hence these polynomials are such that every factor is rotational, including all irreducible polynomials of degree $3k$; also every rotational polynomial of degree $3k$ is a factor of one of these.

We thus have a perfect means of counting irreducible rotational polynomials which, incidentally, all produce forests or tessellations with a period dividing $2^{2k}+2^k+1$ and row-period $m = 1$, since $t+1$ is always a remainder in the development of $1/f^*(t)$ as a sequence.

We also see that t^2+t+1 is the *only* reflexive rotational polynomial that is irreducible. Any such polynomial must be a factor of both $t^{2^k+1}+t+1$ and $t^{2^k+1}+t^{2^k}+1$, and t^2+t+1 is the only possible common factor.

Finally we note that, since reflexive irreducible polynomials divide $t^{2^j+1}+1$, the non-reflexive polynomials that form triplets with all of these, and t^2+t+1, are factors of the polynomials $t^{2^j+1}+t^{2^j}+t$ and $t^{2^j}+t+1$; which form a triplet with $t^{2^j+1}+1$. These characteristic polynomials divide the companions of the reflexive polynomials into two sets having a right-handed and left-handed association. Rotational irreducible polynomials can be similarly subdivided, as a whole. General irreducible polynomials, however, can only be similarly aggregated into pairs of sets of three.

5.3. *Enumeration of reflexive and rotational polynomials*

We have two ways of counting these polynomials, and it is instructive, and an added proof of completeness, to use both.

We can count factors of the polynomials we have shown to be products of the irreducible polynomials, but we shall first use the known totals $I_2(k)$ of primary reflexive and rotational polynomials of degrees $2k$ or $3k$, and the known absence of triangular polynomials of degree $6k$, by eliminating primary composites.

Let $\bar{R}(2k)$ be the number of primary reflexive polynomials of degree $2k$, and let $R(2k)$ of these be irreducible. The reflexive polynomials of degree $2k+1$ are all composite, except for $t+1$, which will not be in our count.

Let $\bar{S}(3k)$ be the number of primary rotational polynomials of degree $3k$, and let $S(3k)$ of these be irreducible. For degrees $3k+1$ and $3k+2$ all are composite, except for t^2+t+1, which will not be in our count (though it is in $R(2)$).

Let $\bar{T}(6k)$ be the number of primary reflexive rotational polynomials of degree $6k$, and suppose $T(6k)$ are irreducible. For other degrees, all are composite, as we shall also show for $n = 6k$. Again t^2+t+1 is excluded.

Let $U(k)$ be the number of irreducible polynomials that are members of *non-reflexive* triplets, there will also be $2R(k)$ non-reflexive polynomials forming triplets with reflexive polynomials, a pair for every such polynomial except t^2+t+1, once again. This exception in several cases means that we must take special care in starting recurrences, when k is small.

We obtain a primary reflexive polynomial of degree $2k$ from the product of the triplet companions of every reflexive polynomial of degree k, $R(k)$ in all. We also have one for each reciprocal pair counted in $S(k)$ and in $U(k)$. There is also a contribution from $T(2k)$, when $3|k$.

We have a primary rotational polynomial of degree $3k$ for every triplet, including those with a reflexive polynomial, i.e. $R(k)+\frac{1}{3}U(k)$ in all.

Finally we obtain a primary reflexive-rotational polynomial of degree $6k$ for every triplet without a reflexive, i.e. $\frac{1}{6}U(k)$ in all, and also from a triplet containing a reflexive of degree $2k$, in number $R(2k)$, and from a pair of rotationals with degree $3k$, in number $S(3k)$.

We thus have the following equations:

$$\left.\begin{aligned} I_2(k) &= \bar{R}(2k) = R(2k)+R(k)+\tfrac{1}{2}S(k)+\tfrac{1}{2}U(k)+T(2k) \\ I_2(k) &= \bar{S}(3k) = S(3k)+T(3k)+R(k)+\tfrac{1}{3}U(k) \\ I_2(k) &= \bar{T}(6k) = T(6k)+R(2k)+\tfrac{1}{2}S(3k)+\tfrac{1}{6}U(k). \end{aligned}\right\} \tag{5.31}$$

Also
$$I_2(k) = 3R(k)+T(k)+S(k)+U(k). \tag{5.32}$$

We now eliminate $U(k)$, $S(k)$, $R(k)$ to obtain (subject to initial conditions) that

$$2T(6k)-T(3k) = 2T(2k)-T(k). \tag{5.33}$$

The polynomial t^2+t+1, which makes $T(2) = 1$, causes an anomaly. If, however, we ignore the cases $k = 1, 2$, the equation is satisfied, by consideration of particular cases for $k \leqslant 12$, $k \neq 1, 2$, and thereafter, only by having $T(k) = 0$. This confirms the earlier argument.

We then have
$$\left.\begin{aligned} R(2k) &= \tfrac{1}{2}[R(k)+I_2(k)] \\ S(3k) &= \tfrac{1}{3}[S(k)+2I_2(k)]. \end{aligned}\right\} \tag{5.34}$$

We now consider direct enumeration.

Reflexive polynomials of degree $2k$ are factors of $1+t^{2^k+1}$. We ignore the factor $1+t$, leaving 2^k zeros in all. These are shared by the irreducible reflexive polynomials of degree $2k/d$, d odd. Thus
$$\sum_{\substack{d\mid k\\ d \text{ odd}}} 2\frac{k}{d}R\left(\frac{2k}{d}\right) = 2^k \tag{5.35}$$

whence
$$R(2k) = \frac{1}{2k}\sum_{\substack{d\mid k\\ d \text{ odd}}} \mu(d)\,2^{k/d}. \tag{5.36}$$

This satisfies the recurrence relation for $R(2k)$ above exactly, using $R(2n+1) = 0$, that is we ignore $(1+t)$, which gives $R(1) = 1$.

For rotational polynomials, we may consider $1+t^{2^k}+t^{2^k+1}$, first removing the factor $1+t+t^2$ when k is even. The number of zeros remaining is then $2^k-(-1)^k$. The zeros are shared between polynomials of degree $3k/d$, $3 \nmid d$ giving

$$\sum_{\substack{d\mid k\\ 3\nmid d}} \frac{1}{2}\frac{3k}{d}S\left(\frac{3k}{d}\right) = 2^k-(-1)^k \tag{5.37}$$

whence
$$\tfrac{1}{2}S(3k) = \frac{1}{3k}\sum_{\substack{d\mid k\\ 3\nmid d}} \mu(d)\,(2^{k/d}-(-1)^{k/d}). \tag{5.38}$$

This satisfies the recurrence for $S(3k)$ throughout, using $S(3k+1) = S(3k+2) = 0$, thus ignoring t^2+t+1, which would give $S(2) = 1$.

Table 3 on page 613 gives $I_2(k)$, $P_2(k)$, $R(2k)$, $\frac{1}{2}S(3k)$ for $k = 1(1)\,30$.

5.4. *The identification of individual polynomials*

This can be rather nicely carried out by use of an additive vector basis for reflexive or rotational polynomials of the appropriate degree. As an example we shall obtain rotational polynomials of degree 18 in this way. We choose a basis for the 2^7-1 polynomials that has 0, 1, 2, 3, 4, 5, 6 zeros at the ends of its seven vectors, and which have as many of the smaller rotational factors as can be arranged. The choice is exhibited in the table below

	basis	rotational polynomials of degree 18	residues mod b	residues mod c	residues mod c'
1	0 0 0 0 0 0 1 0 1 0 1 0 1 0 0 0 0 0 0	a^6	01	011	010
2	0 0 0 0 0 1 1 1 0 0 1 1 0 1 0 0 0 0 0	a^5c	10	0	010
3	0 0 0 0 1 1 1 1 0 0 0 1 1 1 1 0 0 0 0	a^4cc'	01	0	0
4	0 0 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 0 0	a^3c^2c'	10	0	0
5	0 0 1 0 1 1 1 1 0 0 0 1 1 1 1 0 1 0 0	a^2b^3cc'	0	0	0
6	0 1 1 1 1 1 0 0 0 1 0 1 1 0 0 1 0 1 0	ab^3c^2c'	0	0	0
7	1 1 1 0 1 1 0 1 0 1 0 1 0 1 1 0 1 1 1	$b^3c^2c'^2$	0	0	0
$1+7$ with					
—	1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1	$C_{18}(A) = p_6 q_6 q_6'$			
5	1 1 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1	$p_9 p_9'$			
$5+6$	1 0 1 1 1 1 0 0 1 0 1 1 1 0 0 1 0 0 1	$C_{18}'(B) = r_6' s_6' t_6'$			
6	1 0 0 1 0 0 1 1 1 0 1 0 0 1 1 1 1 0 1	$C_{18}(B) = r_6 s_6 t_6$			
$6+4$	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1	p_9^2			
4	1 1 1 1 1 1 0 0 0 1 0 1 1 0 0 1 1 1 1	$p_{18}(C)$			
$4+5$	1 1 0 1 0 0 1 1 0 1 0 0 0 1 1 1 0 1 1	$p_{18}(A)$			
$4+5+6$	1 0 1 0 1 1 1 1 0 0 0 1 1 1 1 0 0 0 1	$p_{18}(B)$			
$3+4+5+6$	1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1	$p_9'^2$			
$3+4+6$	1 0 0 0 1 1 1 1 0 0 0 1 1 1 1 0 1 0 1	$p_{18}'(B)$			
$3+4$	1 1 1 1 0 0 1 1 0 1 0 0 0 1 1 1 1 1 1	$p_{18}'(C)$			
$3+4+5$	1 1 0 1 1 1 0 0 0 1 0 1 1 0 0 1 0 1 1	$p_{18}'(A)$			

Using residues modulo b, c, and c', we see that vector 7 must occur in the representation of any polynomial of full degree 18; vector 1 must occur if there is to be no factor c, and then 2 cannot occur if c' is not to be a factor. With $1+7$, we cannot have 3 without 4 if b is not to be a factor. We may therefore have $1+7$ and any combination of 5 and 6 (including absence) with neither 3 nor 4, or with 4 alone, or with both 3 and 4. The resulting vectors are listed in the table above.

It is of interest to note certain unexplained regularities about this construction. All the factors of $1+t+t^{65}$ contain the vector 4 but not 3; this includes p_9^2, and $p_9 | 1+t+t^{65}$. Likewise all the reciprocal factors have vectors 3 and 4. These vectors are absent from the composite vectors, both primary and $p_9 \times p_9'$.

Similar groupings have been found with $k = 21$ and 24. In none of these cases was there any failure in allocation of the irreducible rotational polynomials, for $k = 18$, 21 or 24, to the correct one of the polynomials $1+t+t^{2^j+1}$ and $1+t^{2^j}+t^{2^j+1}$ *before* the correspondence was actually tested by division. A proof of this connection would be useful.

Reflexive and rotational polynomials are listed in tables 4 and 5 .The reflexive polynomials may be extracted, with effort, from the tables of Marsh (1957), which extend to $k = 19$. They were, however, taken from a table, much easier to use, prepared in Bergen, by S. Mossige (1972) under the direction of E. S. Selmer. This extends to $k = 20$. Rotational polynomials have been identified in the manner described above. The periods of the polynomials are given, and, where available, the row-periods of corresponding tessellations: always $m = 1$ for rotational irreducible polynomials.

Table 4. Irreducible reflexive polynomials

k		n	m
2	111	3	1
4	11111	5	3
6	1001001	9	7
8	111010111	17	5
	100111001	17	15
10	11111111111	11	31
	10010101001	33	31
	11000100011	33	31
12	1111111111111	13	63
	1010011100101	65	21
	1000111110001	65	63
	1011101011101	65	63
	1101011101011	65	63
14	100111111111001	43	127
	101010010010101	43	127
	110100010001011	43	127
	100001010100001	129	127
	100010111010001	129	127
	101001111100101	129	127
	110010111010011	129	127
	111001111100111	129	127
	111111010111111	129	127
16	10001100100110001	257	
	10010000100001001	257	
	10011010101011001	257	
	10101100100110101	257	
	10110111111101101	257	
	10111010101011101	257	
	10111101110111101	257	
	11000000100000011	257	
	11010001110001011	257	
	11010110101101011	257	
	11011011111011011	257	
	11100000100000111	257	
	11101010101010111	257	
	11110001110001111	257	
	11110110101101111	257	
	11111100100111111	257	

k		n
18	1111111111111111111	19
	1000000001000000001	27
	1011100001000011101	57
	1111011101011101111	57
	1000101101011010001	171
	1001110111110111001	171
	1011001101011001101	171
	1011011001001101101	171
	1101000101010001011	171
	1110011101011100111	171
	1000001111111000001	513
	1000010101010100001	513
	1001100011100011001	513
	1010000011100000101	513
	1010010111110100101	513
	1010011001001100101	513
	1010111011101110101	513
	1011111011101111101	513
	1100100111110010011	513
	1101011111111101011	513
	1101100111110011011	513
	1101101001001011011	513
	1110010011100100111	513
	1110101011101010111	513
	1110110001000110111	513
	1111001001001001111	513
	1111100101010011111	513
	1111110001000111111	513

6. The symmetry of tessellations

Before attempting to construct a rotational tessellation by means of the rotational copses it contains, we must investigate the kinds of symmetry that a tessellation can have. If it has none but the basic period-translations we call it a U-tessellation.

The simplest further symmetry is reflexion in a line perpendicular to the ground-line. The

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

periodicity n along this ground-line shows that identical lines of reflexion occur with period n and, clearly, there is a second set of lines of reflexion, different in detail, half-way between each pair of neighbours in the first set. These R-tessellations have reflexive symmetry alone.

If two lines of reflexion occur at an angle, it has to be 60° or 120° for a triangular lattice, and we have triangular symmetry. We can likewise have the rotational symmetry of a triangle, without the reflexion, and we consider this first.

Table 5. Irreducible rotational polynomials

(Factors of $110^{3\kappa-2}1$, of degree $k = 3\kappa$ for $3\kappa = 3(3)\ 27$, and of their reciprocal polynomials, i.e. of $10^{3\kappa-2}11$ for $3\kappa \leqslant 21$. Also $k = 2$.)

k		n
2	111	3
3	1101	7
	1011	7
9	1100000001	73
	1000000011	73
12	1100101011111	273
	1111101010011	273
15	1001001100110111	1057
	1111100011100001	1057
	1110110011001001	1057
	1000011100011111	1057
18	1111110001011001111	1387
	1010111100011110001	4161
	1101001101000111011	4161
	1111001101000111111	1387
	1000111100011110101	4161
	1101110001011001011	4161
21	1011011111100001000111	2359
	1000101010010011000011	16513
	1001011011011011111011	16513
	1010101110101001111111	16513
	1100011101101101110001	16513
	1110011001010111001101	16513
	1110001000011111101101	2359
	1100001100100101010001	16513
	1101111101101101101001	16513
	1111111001010111010101	16513
	1000111011011011100011	16513
	1011001110101001100111	16513

k		n
24	1011001001011011010001101	3133
	1100000100000000101111111	9399
	1000111011101101110110001	21931
	1111010010000000011010011	21931
	1001011000111110000111001	65793
	1001111100001000010101001	65793
	1011101101101101000011101	65793
	1100100000110110111101111	65793
	1101100111010011011110111	65793
	1110010101100101111001011	65793
27	1000010110001011001000101111	262657
	1000100100000000100001111111	all
	1001001000101011011111000111	
	1001010011111101110010100111	
	1001111010100000110110010111	
	1010011000100110100101100011	
	1010101010101101001100110011	
	1011011101010000011111101011	
	1011101111011011110110111011	
	1011110100001101011011011011	
	1100000100010110010110000101	
	1100011111000000111011100101	
	1100110110011101111111010101	
	1100101101001011010010110101	
	1101000001100000101100001101	
	1101011010110110000001101101	
	1110100011100110111111111001	
	1110111000110000010010011001	
	1111100110010000000101110001	

Let A be one centre of rotational symmetry: there are clearly others of the same type A, one to each cell. Suppose B is another centre of symmetry of any type as near as possible to A, i.e. none is nearer, though others may – and will – be equally near. We shall show that B is *not* the same kind of centre as A, i.e. that there is no symmetry of the whole tessellation that moves A to B.

In the diagram the symmetry round A implies three copies of B, as indicated, with BAB′ = B′AB″ = B″AB = 120°. Mark the line AB with a 'flag' to distinguish between its ends and sides. The lines AB′, AB″ carry similar flags by symmetry (see diagram 4). Now, if B is equivalent to A it must also have lines and flags equivalent to AB at A. One of these must go to A or to C: it cannot go between A and C or C″, for that would contradict the assumption that B is a nearest symmetry centre to A. But neither can it go to A, for this would give two flags and 180° symmetry of rotation to AB; such symmetry is not possible, for forest growth is evidently one-sided, with unique development upward from a ground line, and multiple development downwards. Nor can the flag go to C, for this would imply a triangle of flags AB, BC, CA that are equivalent, and hence the centroid of this triangle would have to be a centre of symmetry nearer to A than B.

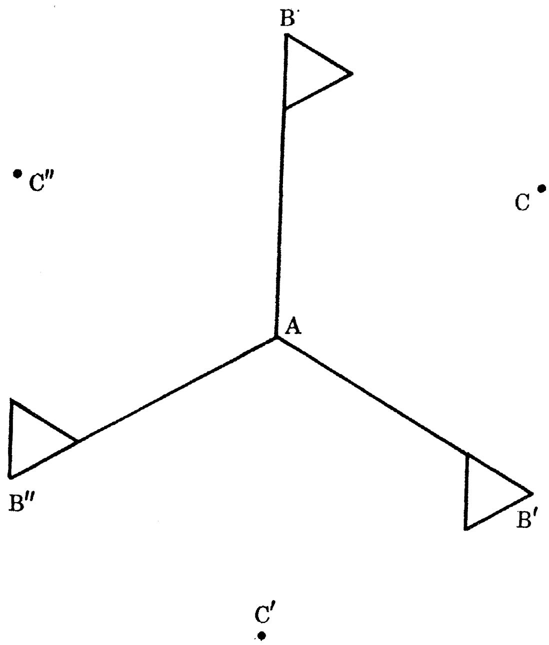


DIAGRAM 4.

So A and B, and similarly C, are three centres of symmetry distinguishable from one another. The whole picture can now be completed with a lattice of symmetry centres forming equilateral triangles of two orientations or kinds, ABC clockwise round one kind of triangle, counter-clockwise round the other. A pair of such triangles forms a cell of the tessellation. The sides of the triangles may be parallel to the ground lines, perpendicular to them, or skew. Such tessellations are called S-tessellations if they are without reflexive symmetry. We note also that lines like B″AC contain centres of symmetry at minimum distance apart, repeating cyclically.

We can also have triangular symmetry or T-tessellations with reflexion as well as rotation. The lines of reflexion are either (*a*) along the lines of nearest centres mentioned above, e.g. bisecting BB′ at right angles, or (*b*) along lines forming one kind of centres only, say A, also as near as possible, and bisecting, for example BC at right angles, so that C is a mirror image of B.

For a rotational tessellation we have $n_1 = n_2 = n_3 = n$, so that the T-period, T, is equal to n also. The S-period has then only two possible values $S = n$ or $S = \frac{1}{3}n$ if $3|n$; any other value can be seen to imply a row period $< n$. The two values of S correspond to the two varieties of T-tessellations mentioned above; for $S = n$ we have version (*a*), with lines of reflexion along the medians of the T-triangle (of side $T = n$), meeting all three types of centre of rotation, all consisting of T-copses, in turn; for $S = \frac{1}{3}n$ we have version (*b*) with lines of reflexion along the medians as before, but meeting only one kind of centre of rotation, which will consist of T-copses, the other centres are at the centroids of triangles of centres of the first kind, and are centres of rotation only, consisting only of S-copses, the two kinds of these being mirror images. We call these two varieties T_3- and T_1-tessellations, the suffix indicating the number of distinct centres involving T-copses.

The centres of rotation each occur in one of three possible sets of positions in the background; the three types of centre are B-centres at the centroids of B-triangles, C-centres at the centroids of C-triangles, and D-centres on a node or dot. It is clear, by considering the long diagonal of a rhombus of which one half is a T-triangle, and on which all three kinds of centre appear at equal intervals, that there is a set of centres of each type B, C or D if $3 \nmid n$, but if $3|n$, we have three sets of distinct kinds of centres of the same type, which may be any of B, C, or D for a particular n ($n = 3, 6, 21$ excepted). T_1-tessellations can exist only if $3|n$, but T_3-tessellations can exist if $3 \nmid n$ or if $3|n$; $n = 63$ gives the first case where both types occur for the same n.

Finally we note that a *glide reflexion* is not possible in a tessellation except as a sub-symmetry of full reflexive symmetry and translation. A glide reflexion implies that, after l rows, say, the original sequence is repeated in reverse, and so, after $2l$ rows, it is repeated in the original sense *with matching values* of $x = r + \frac{1}{2}s$. However, the fact that m and n are exactly divisible by the same power 2^j also implies the existence of an S-period (see § 2.4) and also that $2S$ is the first repetition with matching values of κ. We deduce that l is also a normal row-period, and so the repeated sequence is symmetric, and the glide reflexion is just part of a normal reflexion plus translation. It is of interest to note that either repeated alternation, and/or the production of successive layers (see Miller 1970, §§ 4.6 and 4.7) *can* result in the reversal of a tessellation, and so of its constituent sequences, without full reflexion.

7. The construction of rotational tessellations

We can now set out to construct a rotational tessellation by placing similar rotational copses of suitable size at all or some of the centres of symmetry of one of the three sets of one kind. A rotational tessellation of period n may have any or all of the three types of copses in its make-up; the three types B, C, D give rotationl copses of sizes $3\lambda + \epsilon$, $\epsilon = 3, 2, 1$ respectively, each for all non-negative integral λ, but only *one* set contains n. We may thus need to use copses of size $n+1$ or $n-1$ instead of, or as well as n in building tessellations. We use one of these three sizes at a time, and fit together identical copses in rows parallel to the ground. We may not use all the symmetry centres of the chosen kind; we use a set $[r_c + in, s_c + jn]$ forming a lattice of double equilateral triangles – rhombuses with short diagonal parallel to the ground line. If an $(n+1)$-copse contains $rC = rn^2$ elementary triangles, so that the rhombus of our lattice contains r cells, we use one centre in r.

For copses of size n, each has its base adjacent to that of the next, so that each node of the ground line belongs to just one copse. The next line of copses above has its bases similarly spread along the line $s = n$, so that the vertices of three adjacent copses form a unit B-triangle as in diagram 5. If one corner is at $[r_0, s_0]$ the corresponding corners of other copses are at $[r_0 + in, s_0 + jn]$ and the centre $[r_c, s_c]$ of the first copse is at $[r_0 + \frac{1}{3}(n-1), s_0 + \frac{1}{3}(n-1)]$.

If we use copses of size $n+1$, we place them so that each shares a corner node with two others. This is legitimate, since rotational copses have all corners similarly tagged.

For copses of size $n-1$, we place the bases along the ground and along lines $s = jn$, with *one* node between each pair that belongs to neither copse. Likewise the top node of a copse, on line $jn-2$, is two units from corners on the neighbouring copses above. The three adjacent vertex nodes form a double-sized B-triangle. This has a unit C-triangle in it consisting of the intervening nodes mentioned above; these three must be alike from symmetry, and sum to 0 mod 2, that is *they must be zeros*.

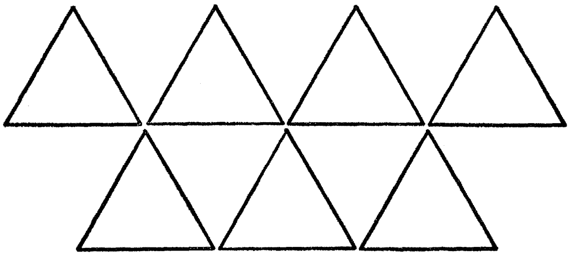


DIAGRAM 5

We have now to choose a copse to which we can apply the treatment outlined above. Not every one will produce a tessellation. Two criteria must be applied.

(1) The row-period n, with the polynomial $\sigma^*(t)$ chosen, must be capable of providing T-period n, so that the next row of copses is correctly developed by the rule (2.14).

(2) The polynomial $\sigma^*(t)$, given by the base n-vector (whether from an n-copse, or an $(n+1)$-copse with the last digit suppressed, or from an $(n-1)$ copse with an extra zero) must contain suitable factors; these are $(1+t)^{2^j}$ where $2^j \| n$, and a factor needed so that $f^*(t)$ is a rotational polynomial where
$$\frac{\sigma^*(t)}{1+t^n} = \frac{\phi^*(t)}{f^*(t)}.$$

If these criteria are satisfied the copses will fit automatically into a purely periodic rotational tessellation. That the empty spaces are rotationally filled in is easy to verify by working inwards symmetrically from the corners of the spaces.

For the first criterion a row-period n is not sufficient, the phase of the repetition must also be right. Consider $n = 21$. The polynomial 100011 gives $n = 21$, $n = 1$ and $T = 21$, and it generates just one rotational tessellation (figure 46). The reciprocal polynomial gives another (a mirror image), and their product yields one fully symmetric T_3-tessellation (figure 47). On the other hand, the polynomial 1010111 has $n = 21$, $m = 3$ but the repetition in row 21 is not in the right place, and no rotational tessellation results.

For the second criterion the basis of §4.4 for any n allows us to pick out vectors divisible by $(1+t)^{2^j}$ very easily for $(n-1)$-copses and n-copses, and, except in one case, for $(n+1)$-copses, since the vectors are all rotational, and the factors $(1+t)^i$ exhibited. For $n = 2^i - 2^j$, $i > j+1$, this is all we need, for all factors of $1+t^n$, except $1+t$, combine into primary rotational polynomials. For other values of n, e.g. $n = 2^{2j} + 2^j + 1$, only some of the factors may combine into rotational polynomials. If the product of all these is $f^*(t)$, always of even degree, $2k$, and if $f^*(t)\,F^*(t)$ is $1+t^n$, then we know that $F^*(t)$ produces an $(n+1)$-copse, with k zeros at each end, since $f^*(t)$ is rotational, and $F^*(t)\,t^k$ is a sum of basic rotational $(n+1)$-vectors. Bases for copses of size n and $n-1$, and further copses of size n may be obtained by multiplying $F^*(t)$ by $b = 111$ and b^2, and all three by c, b^3, b^3c, etc. in the usual way, since $F^*(t)$ is a sum of basic vectors derived from rotational polynomials, and so are all the new vectors.

This process gives exactly the right number of vectors. It terminates with an $(n+1)$-vector having units at both ends, and this requires special treatment, for the final unit belongs to the subsequent n-period, and so is omitted from $\sigma^*(t)$, and it is $\sigma^*(t)$ which must satisfy the second criterion.

Now, examination of table 2 and counting digits shows that the only vector of the basis that has 1's at the ends is the one at the top of the second group for any n; this is always part of the longest basic vector so far produced to be used in the basis for copses for building tessellations. In an n-copse, this vector is completely valid: it is always divisible by at least $(1+t)^{2^j}$ and by $F^*(t)$. If combined with other vectors, they are also divisible by the desired $(1+t)^{2^j}$. However, it is easily seen from this table that, for the corresponding $(n+1)$-vector, the power of $(1+t)$ is one unit short in many cases. For example for $n = 24$ the polynomial $b^6c(1+t)^8$ is usable, but the 25-vector for $b^7c(1+t)^7$ is not divisible by $(1+t)^8$, as is needed. For $n = 42$, $b^8c(1+t)^{22}+b^6c(1+t)^2$ is divisible by $F(t) = (1+t^{42})/b^2c^2c'^2$ but, for the 43-vector, the polynomial $b^9c(1+t)^{21}+b^6c(1+t)$ lacks a factor $(1+t)$. More seriously, the $(n+1)$-vector must be deprived of its final digit because of superposition, so far as its factorization is concerned, but not with regard to its rotational properties. What we need, and can obtain, is a vector to add to it which, *without* its final digit, is rotational, but *with* the final digits is a multiple of $F^*(t)/(1+t)$. When combined *without* the final digit of the second vector, the sum will be rotational, but with the final digit, it will be divisible by $F^*(t)$ since both parts are *exactly* a factor $(1+t)$ short. The vector needed is $(0\ 1^{n-1}0)$ or $\{(1+t^n)/(1+t)\}-1$. This is rotational for all n, being obtained by attaching a row all round the zero $(n-2)$-copse, and $(1+t)^n/(1+t)$ is clearly a multiple of $F^*(t)/(1+t)$; this vector has a zero at each end, and uses the rejected part of the basis. Thus we have constructed a *usable* vector of full length, and this is always sufficient.

Rotational tessellations exist for all $n = 2^i-2^j$, $i > j+1$, generated by rotational factors of $1+t^n$. They exist also for $n = 2^{2j}+2^j+1$, generated by factors of $1+t^{2^j}+t^{2^j+1}$, and for some sub-multiples of these periods, where a factor has a period shorter than the full one. L.c.ms of these periods also have rotational tessellations. Examples of shortened periods are $n = 85 = \frac{1}{3}(2^8-1)$ given by

$$111010111 \times 110111101 \times 101111011,$$

and $n = 1387 = \frac{1}{3}(2^{12}+2^6+1)$ given by 1111001101000111111 and by its reciprocal, and $n = 3133 = \frac{1}{21}(2^{16}+2^8+1)$.

8. The enumeration of rotational and triangular tessellations

8.1. *Clearings*

One of the most convenient ways to enumerate and to list distinct tessellations is by means of clearings. A clearing is a triangular cluster of vacant nodes surrounded by live nodes: the inside of an alternate-sided hexagon in a tessellation. Its size is measured by the length of the longest line of consecutive vacant nodes, or *gap*, it contains.

Clearings were used in Miller (1968) to provide a useful means for identifying and listing distinct tessellations in the set generated by a polynomial $f^*(t)$. S. Golomb (see Selmer 1966 p. 169) showed that, if $\deg f^*(t) = k$, then, amongst all the cycles generated there are *gaps*, successions of zeros sandwiched between two 1's, of lengths as follows:

size of gap	$k-1$	$k-2$	$k-3$	...	$k-i$	...	3	2	1	0
number of gaps	1	1	2	...	2^{i-2}	...	2^{k-5}	2^{k-4}	2^{k-3}	2^{k-2}

A gap of length 0 is just a pair of 1's.

From this we can enumerate clearings, since a clearing of size r contains sequences with every gap from 0 to r in its successive rows, thus

size of clearing	$k-1$	$k-2$	$k-3$	$k-4$	$k-5$	…	$k-i$	…	1	0
number of clearings	1	0	1	2	4	…	2^{i-3}	…	2^{k-4}	2^{k-3}

A clearing of size 0 is a unit B-triangle of live nodes.

These results can be modified to give the numbers of clearings in purely periodic tessellations, of symmetrically placed clearings in reflexive tessellations, and in rotational tessellations, and of clearings with full triangular symmetry in T-tessellations.

8.2. *Clearings in purely periodic and symmetrical tessellations*

We have first to confine attention to vectors usable in purely periodic tessellations. Those in tessellations generated by $f^*(t)$, which must have $(f^*(t), 1+t) = 1$, are all multiples of $F^*(t)$, where $f^*(t)\,F^*(t) = 1+t^n$, with $n = \text{per}\,(f^*(t))$; thus if $(1+t)^{2^j} \| (1+t^n)$ then $(1+t)^{2^j} \| F^*(t)$ also. A basis for the n-vectors formed by the coefficients in these multiples, assumed of formal degree $n-1$, is given by the polynomials

$$F^*(t), tF^*(t), t^2F^*(t), \ldots, t^{k-1}F^*(t)$$

where $k = \deg\,(f^*(t))$.

We now count sequences originating with a live node at the edge of a clearing, i.e. just after the end of a gap of the clearing, and including the final zero gap where the edges meet in a pair of 1's. These sequences all have the first cycle beginning with a one and so *must include the first vector of the basis*, given by $F^*(t)$. This excludes all vectors that start in a gap.

Thus the gap at the end of the first period in any such sequence is the full gap in a vector across that clearing, and its length depends only on the vector $t^iF^*(t)$ of the basis with largest i that is used in its formation. The length of the gap is thus $k-i-1$. Thus the number $N(r)$ of usable vectors with gap of length r at the end, and the corresponding number $C(r)$ of clearings are as follows:

r	$k-1$	$k-2$	$k-3$	$k-4$	…	$k-i$	…	1	0
$N(r)$	1	1	2	4	…	2^{i-2}	…	2^{k-3}	2^{k-2}
$C(r)$	1	0	1	2	…	2^{i-3}	…	2^{k-4}	2^{k-3}

Now consider reflexive tessellations. We use a basis of reflexive sequences which end at a node on a line of symmetry. This node must be vacant and so the middle of a gap of odd length. We still want a basis of multiples of $F^*(t)$, now self-reciprocal, like $f^*(t)$. The multipliers of $F^*(t)$ are the set of reflexive polynomials of degree $< k = 2\kappa = \deg\,(f^*(t))$. We want only those vectors with a zero as last element, at a centre of symmetry. The vectors we may use as a basis are coefficients in the polynomials

$$t^{\kappa-1}F^*(t), \quad t^{\kappa-2}(1+t^2)\,F^*(t), \quad \ldots, \quad t(1+t^{2\kappa-4})\,F^*(t), \quad (1+t^{2\kappa-2})\,F^*(t)$$

with $\quad 2\kappa-1, \quad 2\kappa-3, \quad \ldots, \quad 3, \quad 1$

zeros in the gap.

Again, the size of the gap in a vector depends on that basic vector used in its makeup having the smallest gap. We find $N_R(r)$, $C_R(r)$, numbers of sequences and clearings with gap or size r as follows:

r	$2\kappa-1$	$2\kappa-3$	$2\kappa-5$	...	$2\kappa-2i-1$	...	3	1
$N_R(r)$	1	2	4		2^i		$2^{\kappa-2}$	$2^{\kappa-1}$
$C_R(r)$	1	1	2		2^{i-1}		$2^{\kappa-3}$	$2^{\kappa-2}$

For rotational tessellations, the vector bases already set up will serve to count clearings. In constructing tessellations, copses were placed in such a way that their vertices were as near a possible to neighbouring symmetry centres. An $(n+1)$-vector or copse has its end or vertex on a D-centre; an n-vector or copse at a corner of a B-triangle round a B-centre, and a $(n-1)$-vector or copse has the intervening zero between vectors at the vertex of a C-triangle round a C-centre. In each case the number of zeros at either end of the vector determines the size of the clearing concerned.

An $(n+1)$-vector with r end-zeros gives a clearing of size $3r-2$

A n-vector with r end-zeros gives a clearing of size $3r$

An $(n-1)$-vector with r end-zeros gives a clearing of size $3r+2$

The vectors we use are, as before, multiples of $F^*(t)$, with $f^*(t)$ chosen to be rotational, and we multiply by rotational polynomials, until the full length of an $(n+1)$-vector is achieved. For this exact length only, a modified vector is needed and, as we have seen, available. We use vectors with one kind of centre at a time, but totals below include all. For each kind of centre, the number of clearings of each size – again dependent on the smallest number of end zeros in any vector of the basis used in its construction – is exactly twice the number for size 3 less, until finally centres which are live nodes, using the modified $(n+1)$-vector of full length, are twice as many as centres in clearings of size 1, which do not use the vector just mentioned, but do use the $(n-1)$-vector with just 1 zero at each end, these end-zeros over-lapping in use.

For period n, then, the numbers of clearings $C_S(r)$ of size r surrounding centres of rotation, and generated by a rotational $f^*(t)$ of degree $k = 3\kappa+\epsilon$, $\epsilon = 0, 1, 2$ are:

r	$k-1$	$k-2$	$k-3$	$k-4$	$k-5$	$k-6$	...	$k-3i-1$	$k-3i-2$	$k-3i-3$	...
$C_S(r)$	1	0	1	2	1	2	...	2^i	2^{i-1}	2^i	...

with the table finish depending on the value of ϵ:

r	3	2	1	0	D	total
$C_S(3\kappa)$	$2^{\kappa-2}$	$2^{\kappa-1}$	$2^{\kappa-2}$	$2^{\kappa-1}$	$2^{\kappa-1}$	$3(2^{\kappa}-1)$
$C_S(3\kappa+1)$	$2^{\kappa-1}$	$2^{\kappa-2}$	$2^{\kappa-1}$	2^{κ}	2^{κ}	$3(3.2^{\kappa-1}-1)$
$C_S(3\kappa+2)$	$2^{\kappa-2}$	$2^{\kappa-1}$	2^{κ}	$2^{\kappa-1}$	$2^{\kappa+1}$	$3(2^{\kappa+1}-1)$

A similar analysis for centres of full triangular symmetry gives, for $\deg f^*(t) = k$,

$k-r$	1	3	5	7	9	11	13	15	17	...	$6i+1$	$6i+3$	$6i+5$
$C_T(r)$	1	1	1	2	2	2	4	4	4	...	2^i	2^i	2^i

Here, as elsewhere, a live centre D behaves like a clearing of size -2.

8.3. *Enumeration of rotational tessellations*

We have now only to use the fact that rotational tessellations each use three rotational centres, and the T_1-tessellations use just one T-centre, while T_3-tessellations use three T-centres.

TABLE 6.

n	all rotational tessellations of period dividing n	rotational tessellations of minimum period n		
		T_1	T_3	S
3	1	1	—	—
6	2	1	—	—
7	3	—	1	2
12	7	3	—	2
14	15	—	2	10
15	31	—	4	26
24	47	12	—	28
28	255	—	12	228
30	767	—	21	714
31	1 023	—	31	992
48	2 047	108	—	1 892
56	65 535	—	240	65 040
60	524 287	—	655	522 860
62	1 048 575	—	992	1 046 560
63	2 097 151	59	1 343	2 095 742
96	3 145 727	4 992	—	3 138 688
112	4 294 967 295	—	65 280	4 294 836 480
120		—	436 220	
21	7	1	—	2
42	47	5	—	22
84	2 047	69	—	1 686
73	63	—	7	56
85	255	—	15	240
93	2 047	—	31	992
105	381	—	4	86

As an example, consider $n = 42$.

Rotational polynomials dividing $(1+t)^{42}$ are $111^2.1011^2.1101^2$. Take $f^*(t) = 111^2.1011^2.1101^2$ with $\deg f^*(t) = 16$, $\operatorname{per} f^*(t) = 42$. The number of rotational clearings is $3(3.2^4-1) = 141$. Of these, 17 are T-clearings. All rotational tessellations of periods 3, 6, 7, 14, 21 are also included. We tabulate thus:

n	3	6	7	14	21	sub-total	42	total	
T_1	1	1	—	—	1	3	5	8	using
T_3	—	—	1	2	—	3	—	3	$8+9$ T-copses
S	—	—	2	10	2	14	22	36	$16+108$ S-copses

Periods up to 21 use $3+3\times 3 = 12$ T-copses, leaving 5 unused, and $2\times 3+3\times 14 = 48$ S-copses, leaving 76 unused.

The tessellations with reflexion are all T_1-tessellations. There are 5 of these, leaving 66 S-copses for 22 S-tessellations.

For $n = 63$ we have one more complication. The polynomial

$$f_1^*(t) = 111 \times 1011 \times 1101 \times (1001001 \times 1011011 \times 1101101)$$

generates tessellations with $n = 63$, $m = S = 21$, and yields T_1-tessellations, but $(1+t^{63})/(1+t)$ also gives rotational tessellations with $n = S = 63$, i.e. any T-tessellations are T_3 if any of the other factors 1010111, 1110101, 1000011, 1100001, 1100111, 1110011 is present in the minimum generating polynomial.

We must therefore count the T_1-tessellations first, and then the polynomial $(1+t^{63})/(1+t)$ may be used as $f^*(t)$ to yield the rest.

The polynomial $f_1^*(t)$ generates 61 T-clearings.

Two of these arise from T_1-tessellations for $n = 3$, $n = 21$. Thus there are 59 T_1-tessellations for $n = 63$. The test of the enumeration is straightforward. The results are given in table 6.

The possible values of n for rotational forests are:

(i) $n = 2^i - 2^j$, $i > j+1$. Any rotational factor of $1+t^n$ will provide rotational tessellations; all factors are included, though some have to occur in complete triplets.

(ii) $n = 2^{2j} + 2^j + 1$. Any factor of $1+t+t^{2j+1}$ or $1+t^{2j}+t^{2j+1}$ will do.

(iii) Occasional submultiples of the above occur with rotational tessellations, e.g. in (i) $n = 85 = (2^8-1)/3$, and any multiple by 2^j, with $f^*(t)$ a power of $111010111 \times 110111101 \times 101111011$; $n = 341 = (2^{10}-1)/3$, and multiples by 2^j, is another; these arise from (i). From (iii) we have $n = 1387 = (2^{12}+2^6+1)/3$, and $n = 3133 = (2^{16}+2^8+1)/21$, and either multiplied by 2^j.

(iv) Any l.c.m. of periods mentioned above.

There are no doubt many others under (iii), but they have, at present to be found by trial.

9. Tables

The tables in this paper are selected from more extensive tables, prepared in the course of these investigations by hand and by computer, often as research exercises by students. It is proposed to lodge the extended tables, sometimes incomplete in parts, in the Library of the Department of Pure Mathematics and Mathematical Statistics in Cambridge University.

Table 1, p. 606, gives numbers of copses of the four symmetry types, and the total numbers of patterns for sizes $n = 1(1)36$; this is the complete table as prepared.

Table 2, pullout, gives a table of bases for rotational vectors for sizes $n = 1(1)50$.

Table 3, p. 613, gives numbers of irreducible and of primitive polynomials, $I_2(k)$ and $P_2(k)$, and the numbers of irreducible reflexive and rotational polynomials, $R(2k)$ and $S(3k)$ for $k = 1(1)30$.

Table 4, p. 618, gives a list of irreducible reflexive polynomials. A complete list is given for $k = 2(2)18$. Values of the period n and row-period m of tessellations generated by these polynomials are also given. This is part of an extended table to $k = 30$, with values of n (only). This was obtained by J. J. B. Parker in 1972 as part of a Diploma Dissertation.

Table 5, p. 619, gives a list of irreducible rotational polynomials. It is given for degree $k = 2$, 3(3)27, together with the period n, (the row-period is always $m = 1$). Again this is part of an extended list to $k = 45$, also obtained by J. J. B. Parker in 1972. For $n \geqslant 24$, only one polynomial of each reciprocal pair is given; those given in this case are all factors of 11000...0001 (with $k-2$ zeros). These factors are given first for $n \leqslant 21$; the others divide 1000...00011.

Table 6, p. 626, gives numbers of all rotational tessellations of period dividing n, together with numbers of rotational tessellations of types T_1, T_3, and S, for various n to 120, thought to be complete to $n = 112$.

We list also several useful unpublished tables that extend and supplement those in Miller (1970), and that have served as auxiliary tables in the investigations. I propose also to lodge these tables in the Department of Pure Mathematics and Mathematical Statistics at Cambridge.

Table 4 in Miller (1970) has been extended, by means of material obtained using a computer program by D. B. Webster, to provide all triplets of polynomials, together with period n and row-periods m, for degree $k = 12$. It is remarkable that, amongst the 53 U-triplets of degree 12 polynomials there are just two comprised entirely of primitive polynomials only, though there are 144 such polynomials in all.

A machine-script factor-table of binary polynomials of degree $\leqslant 14$ has been computed by M. J. T. Guy. There are also other similar tables for GF(p), $p = 3, 5$, etc.

Several tables of irreducible polynomials exist. One of the best was prepared by S. Mossige (1972) under the supervision of Professor E. Selmer, in Bergen, Norway. This gives all irreducible binary polynomials of degree $k \leqslant 20$. For each degree k, the polynomials are given in blocks having the same period n dividing $2^k - 1$; in each block they are in 'lexicographical' order of binary coefficients. Also given is the degree of the zeros of each equation, expressed as a power of a zero of a selected basic primitive polynomial; the period n of each polynomial is thus easily sorted out, and the power-labels provide material for a canon similar to that of Jacobi (1839).

This table is substantially easier to use for our purposes than that of Marsh (1957), which gives straight lists for $k \leqslant 19$. Peterson (1961) gives condensed tables for $k \leqslant 16$, together with supplementary information on periods and linear independence or dependence of the zeros of the polynomials.

10. Illustrations

10.1. *General remarks*

The visual aspect of the copses and tessellations investigated in this paper is of interest in promoting the understanding and appreciation of the subject. Therefore, some care has been taken in choosing illustrations. Several points need to be taken into account.

(1) The use of colour plays an important part in exhibiting symmetries, as can be seen in the colour diagrams in Miller (1970) and ApSimon (1970). Two colour plates are included in this paper which give further illustration to this point.

Colour is even more important with ternary tessellations (postponed to a later paper for further discussion). With such tessellations hexagonal patches of three distinct colours may be arranged and exhibited with complete symmetry of treatment amongst the three colours. The identification of the colours with the integers modulo 3 allows the use of algebra in GF(3) for a study of the existence and numerical properties of these tessellations, but the special qualities of 0 as compared with 1 and 2 must eventually be suppressed, and the use of colour achieves this end satisfactorily.

(2) Symmetry types. We must make sure to exhibit tessellations of all significant symmetry types, bearing in mind that this paper is mainly concerned with rotational symmetry.

The symmetry *types* of copses, U, R, S and T, are described in § 3.2. The three *kinds* of symmetry centre of these copses B, C, and D, are described in § 6. Besides the several combinations of these types and centres, we may also consider the *lattice* of centres of symmetry in any S- or T-tessellation. Lines joining closest centres may include some parallel or perpendicular to the edges of basic unit triangles, or be all skew. Call these H-, V-, or W-lattices. Then:

	symm. type of centre	lattice
A T_3-tessellation has	$3 \times T$	H
A T_1-tessellation has	$1 \times T$, $2 \times S$	V
An S-tessellation has	$3 \times S$	H
An s-tessellation has	$3 \times S$	W

The s-tessellations usually have relatively small cell-size.

The kinds of symmetry centre are not identical in all of these types of tessellation. In any one, if $3 \nmid n$, we always have equal numbers of B-, C- and D-centres. If $3|n$, centres are always all of the same kind, which may be any of B, C, or D.

(3) It is of interest to exhibit a part of all tessellations with small period n, and also with small cell-size $2C = 2mn$, up to reasonable limits in n or C.

However, we do not need to reproduce all diagrams that have already been published elsewhere (maybe in different form, e.g. as forests); some are in colour (ApSimon 1970) where we do not use colour here.

It is possible to take copies of forests from diagrams in Miller (1970) and complete these into tessellations, by inserting horizontal links by hand; we shall consider forests published as equivalent to tessellations for our purposes, with the exception of $T_{15,1}$ to $T_{15,4}$, see figures 42–45, which exhibit T_3-tessellations of each kind having all B, all C, and all D centres.

There is a natural break at $n = 15$, with no further S or T-tessellations until we reach $n = 24$, except for two interesting cases, T_{21} and S_{21}, see figures 47 and 46. We take $n = 21$ as our limit for full coverage of S and T-tessellations.

For T-tessellations we extend coverage to $n = 24$ with 12 tessellations and also give the five T-tessellations for $n = 42$. For $n = 28$, 30, 31 the T-tessellations are rather numerous (12, 21, 31) and we show only a few of the interesting set for $n = 31$.

For S-tessellations, we give all 13 for $n = 15$, and small-celled s-tessellations S_{21}, with $C = 21$, five $S_{42,1}$ to $S_{42,5}$ with $C = 84$, and five of the set of seven s-tessellations $S_{73,1(1)7}$ with $C = 73$ ($S_{73,4}$ and $S_{73,6}$ are given in ApSimon (1970), in his notation as (10, 3, 4, 3) and (9, 6, 1, 4)).

10.2. *Published illustrations*

Before describing the illustrations given in this paper, we make some comments on diagrams already published in earlier papers. Most of these are of forests given in Miller (1970), in which a full list (table 6) is given for $n \leqslant 15$; this list will be used for reference.

In that paper F1 is given in figure 9 as a forest. As a tessellation it is the familiar covering of the plane by regular hexagons, known as the hexagonohedron, and a very familiar one in tiles. As one of our tessellations it has 'too much' symmetry in that the 'direction of growth' seems reversible, in this case only. F2 appears as a tessellation in the centre of figure 8; it is, of course merely reflexive, and not rotational. F4 appears in figures 3 and 13, and as a tessellation in figure 4; this is the first s-tessellation, that is, with minimum n and C. F17 is illustrated in three distinct ways, in colour, in figure 48; one appears as a tessellation in the style used for copses in the colour plates in the present paper.

Other examples of tessellations are given in Miller (1968): F5 is given in fig. 5; the first T_3-tessellation. F32, F33, F34 with $n = 14$ all give the U-tessellation in figure 7. Figure 8 gives the unique smallest-celled U-tessellation (apart from a mirror-image) with $n = 31$, while F26 with $n = 14$ yields the attractive T-tessellation in fig. 9, which is also given in colour as fig. 49 of plate 1 in Miller (1970).

Yet other examples appear in ApSimon (1970), as tessellations in large copse form, and attractively coloured in a way that much enhances appreciation of symmetry. Four of his illustrations are represented in table 6 of Miller (1970), namely:

	Miller (1970)	ApSimon (1970)
6×2	F3 (figure 11)	(2 0 0 1)
5×3	F2 (figure 10)	(2 1 0 1)
14×2	F22 (figure 24)	(3 3 1 1)
12×4	F17 (figure 26)	(6 4 0 3)

Two others are mentioned in the present paper, but are not illustrated here

73×1	$S_{73,4}$	(10 3 4 3)
	$S_{73,6}$	(9 6 1 4)

10.3. *Illustrations of copses*

In this paper we have one sheet, in colour, giving illustrations of copses.

Plate 1 shows copses for $n = 1(1)\,k$, in figures $1(1)\,k$, with more closely restricted scope as k increases.

Figures 1 and 2 show all copses with $n = 1, 2$. For $n = 1$ these are the unit cells, corresponding to vacant (0, yellow) and live (1, blue) nodes. For $n = 2$ we have four arrangements of unit C-triangles, with two patterns under rotation and reflexion.

Figures 3 and 4 show all patterns for $n = 3$ and 4.

Figures 5–9 show all S- and T-patterns for $n = 5$ to 9.

The first copse in each figure is all yellow, the blank or zero background copse.

10.4. *Illustrations of tessellations*

Plate 2 shows some tessellations, in the style used for copses with hexagonal patches of two colours. With table 6 of Miller (1970) again for reference:

figure 10 shows T_3 (F1) 3×1

figure 11 shows S_7 (F4) 7×1

figure 12 shows as above, but mirror image

figure 13 shows R_5 (F2) 5×3

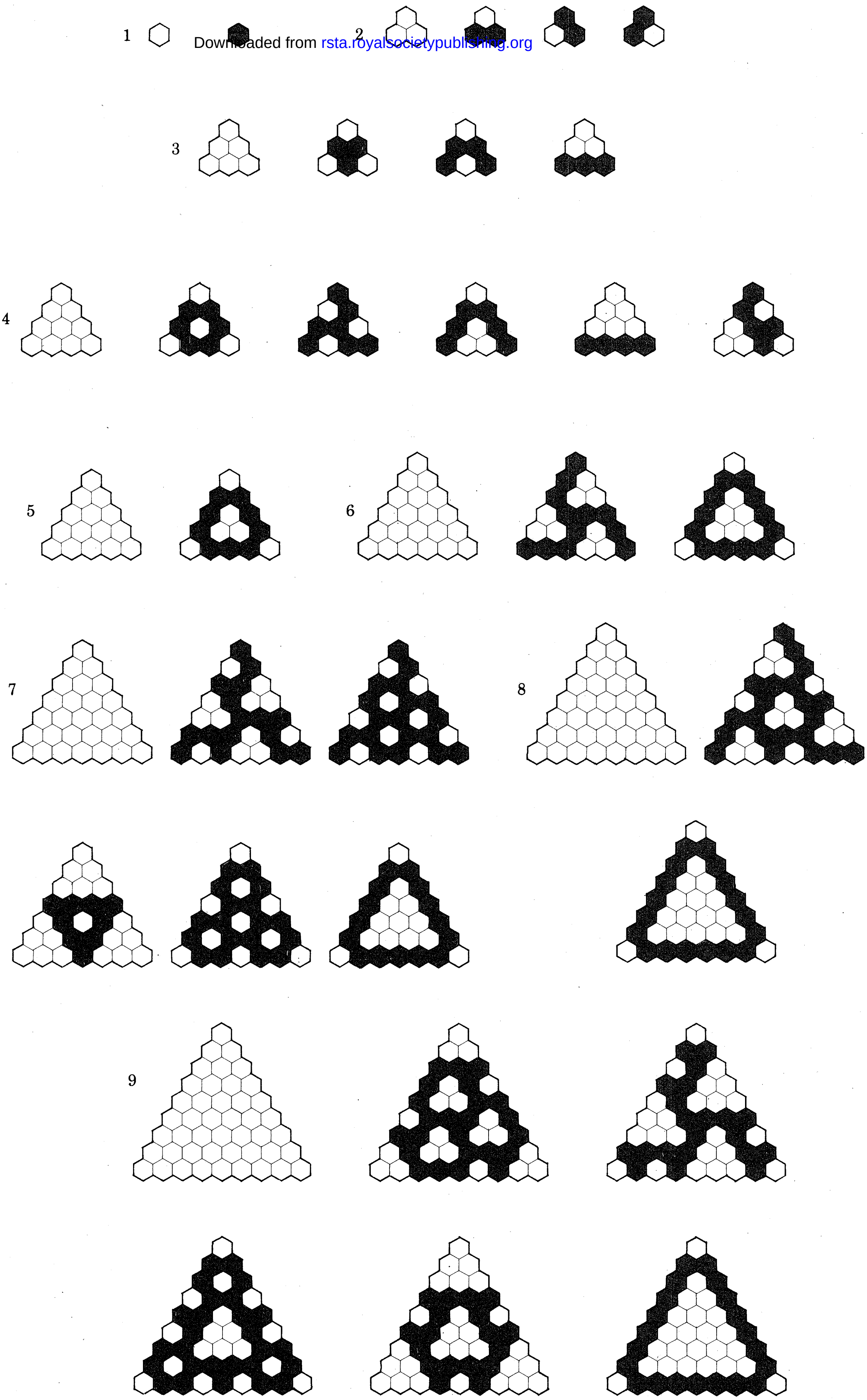
figure 14 shows T_6 (F3) 6×2

figure 15 shows $T_{12,3}$ (F17) 12×4

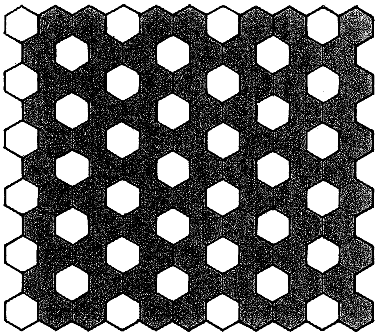
figure 16 shows T_7 (F5) 7×7

The remainder of the illustrations use the representation by joining adjacent nodes that is used in Miller (1970) for forests, with rectangular background areas, in the first few cases, figures 17–23 showing T_3, S_7, R_5, T_6, T_7, $T_{12,3}$, and U_{31}. All the others use the triangular (copse) background area as in ApSimon (1970); table 8 (p. 640) gives further information and the generating fractions.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

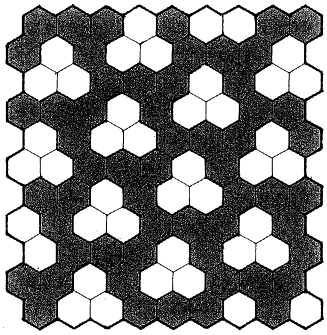


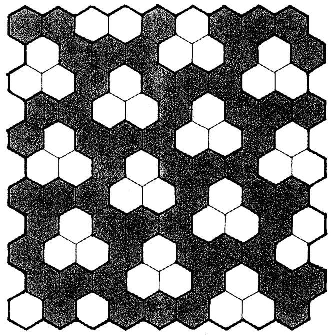
FIGURES 1–9. Copses. For descriptions see §10.3.



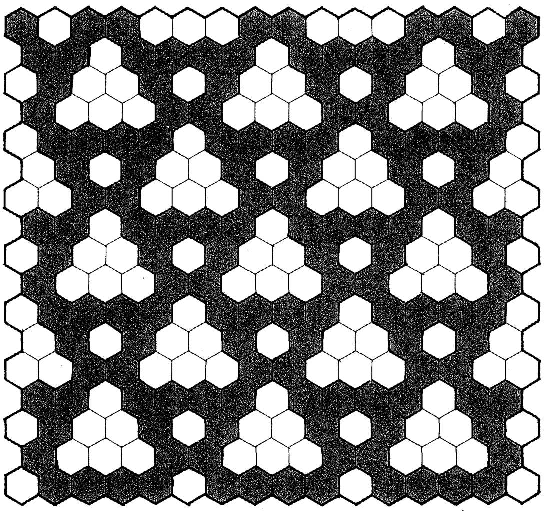
10

11

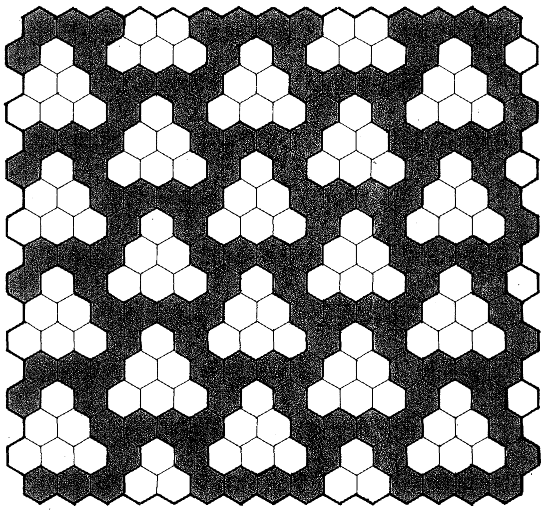




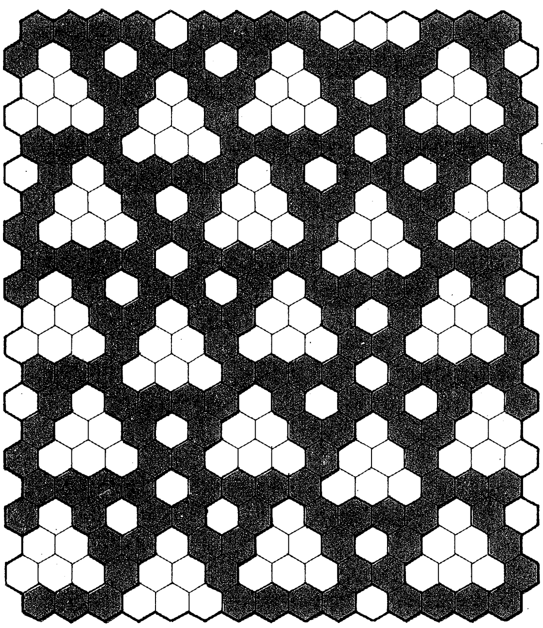
12



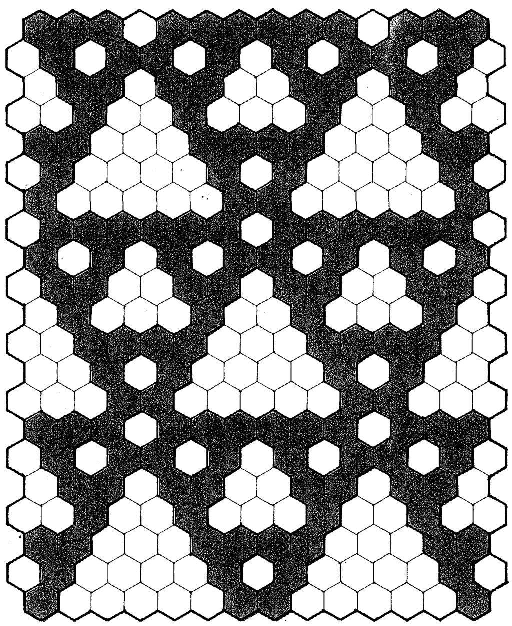
13



14



15



16

FIGURES 10–16. Some simple S- and T- tessellations.
For descriptions see §10.4.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

The latter are also listed in table 6 of Miller (1970), or for $n \geqslant 21$ here in table 7, in a slightly condensed form so that $\varphi(t)/f(t)$ is not always given in its lowest terms, when a common denominator allows condensation. Some extra tessellations, not illustrated here, are also mentioned in table 7. Individual details of the tessellations in figures 17–80 are given in table 8.

TABLE 7.

	m	C	$f(t)$	$\varphi(t)$	figure nos.
T_{21}	7	147	bcc'	1	47
S_{21}	1	21	bc	1	46
$T_{24,1(1)12}$	8	192	b^8	φ	48–59
$\varphi = 1, b, b^2, b^3, cc', bcc', b^2cc', b^3cc', (cc')^2, b(cc')^2, (cc')^3, b(cc')^3$					
$S_{24,1(1)14}$	8	192	b^8	φ	—
$\varphi = c, bc, b^2c, b^3c, c^2, bc^2, c^3, bc^3, c^2c', bc^2c', c^3c', bc^3c', c^3c'^2, bc^3c'^2$					
$T_{28,1(1)12}$	28	784	$(cc')^4$	φ	—
$\varphi = c^4, bc^4, b^2c^4, b^3c^4, c^4C_{12}, bc^4C_{12}, b^2c^4C_{12}, b^3c^4C_{12}, c^3, bc^3, b^2c^3, b^3c^3$					
$S_{28,1(1)114}$	all organized, but not listed here				—
$T_{30,1(1)21}$	30	900	$b^2C_{12}^2$	φ	—
$\varphi = b^{0(1)7}, b^{0(1)7}cc', C_{12}, p_9p_9', bp_9p_9', cc'p_9p_9', bcc'p_9p_9'$					
$S_{30,r}$	not organized or listed				
$T_{31,1(1)31}$	31	961	$(1+t)^{31}/(1+t)$	$b^{0(1)30}$	75–80
$S_{31,r}$	not organized or listed				
$T_{42,1(1)5}$	14	588	$b^2c^2c'^2$	φ	70–74
$\varphi = 1, b, cc', C_{12}, bC_{12}$					
$S_{42,1(1)5}$	2	84	b^2c^2	φ	60–64
$\varphi = b, c, bc', 1, c'$					
$S_{42,6(1)11}$	14	588	$b^2c^2c'^2$	φ	—
$\varphi = c, bc, p_9, bp_9, c'p_9, bc'p_9$					
$S_{73,1(1)7}$	1	73	p_9	$b^{0(1)6}$	65–69
$T_{85,1(1)15}$	85	7225	$C_{24,*}$	$b^{0(1)14}$	—

All the tessellations listed in the table have been drawn, except for $S_{30,r}$ and $S_{31,r}$.

Notation. $T_{24,1(1)12}$ are labels $T_{24,1}, T_{24,2}, \ldots, T_{24,12}$ in order as listed under φ. Likewise $b^{0(1)6}$ means $1, b, b^2, \ldots, b^6$. The numbering of the labels is not quite consistently chosen, but serves for reference.

Individual polynomials used, in detached coefficient notation, are $b = 111$, $c = 1011$, $c' = 1101$, $p_9 = 1000000011$, $p_9' = 1100000001$. These are all irreducible. Also

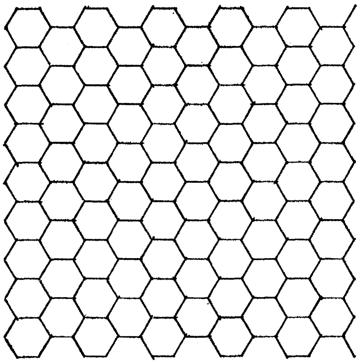
$$C_{12} = 1001001001001 = 10011 \times 11111 \times 11001$$

$$C_{24,*} = 110111101 \times 111010111 \times 101111011.$$

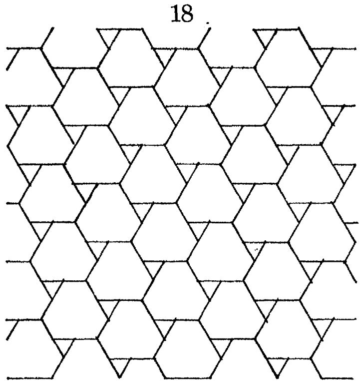
Some numerators $\varphi(t)$ need reduction mod $f(t)$ before use. Finally

$$F_{31,*} = 100101 \times 111101 \times 111011.$$

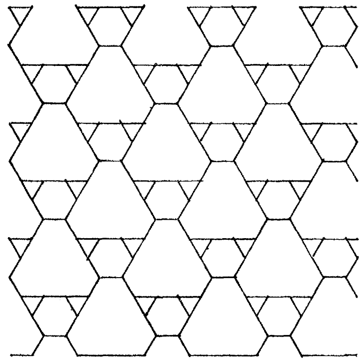
17



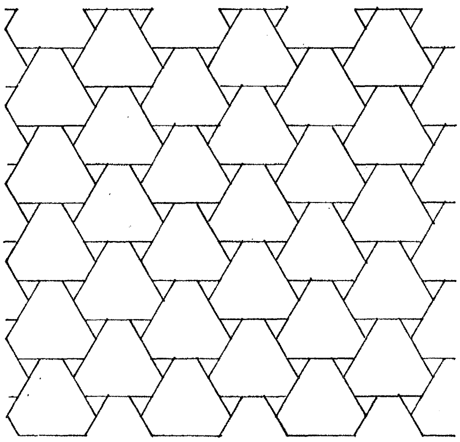
18



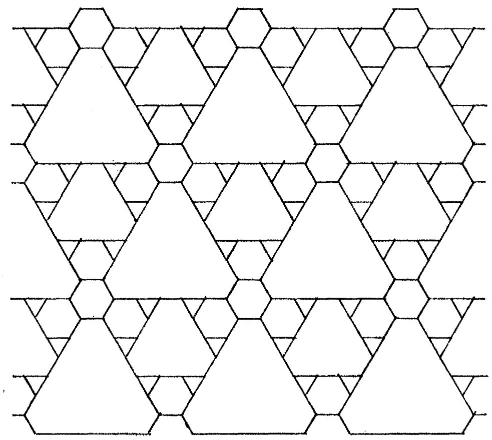
19



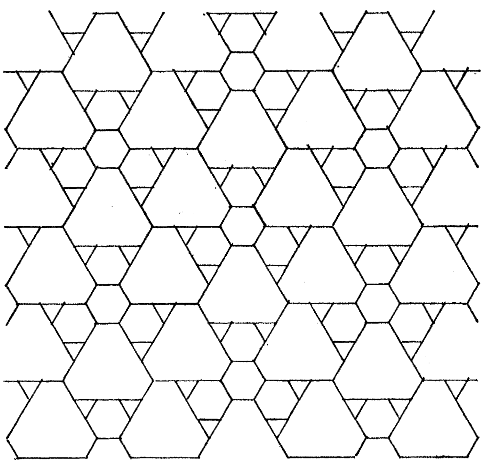
20



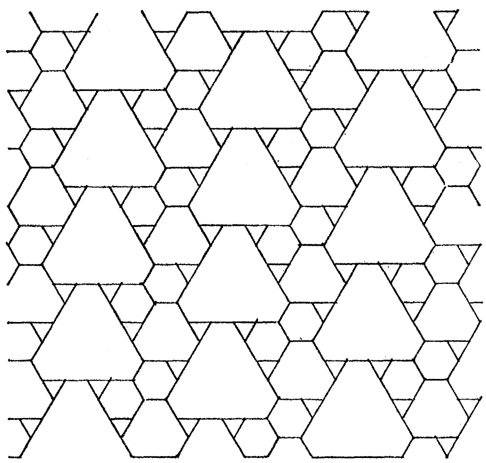
21



22

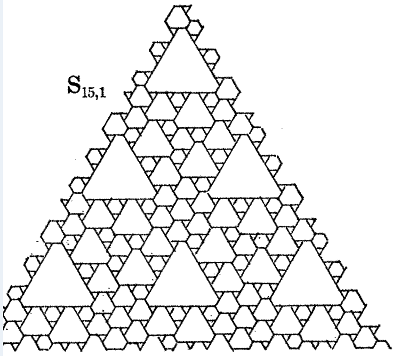


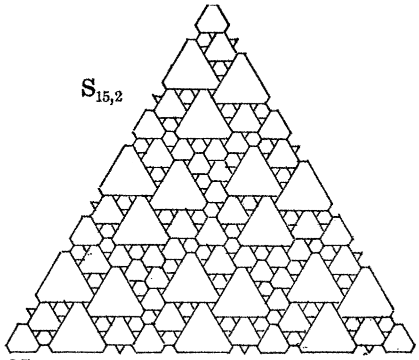
23



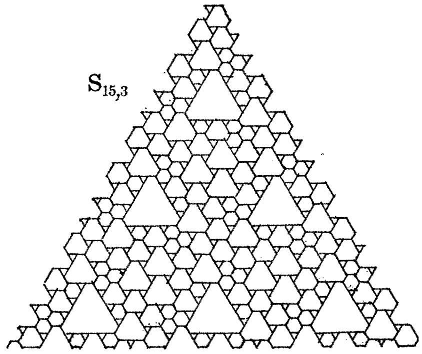
FIGURES 17–23. Parts of tessellations, with rectangular borders.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

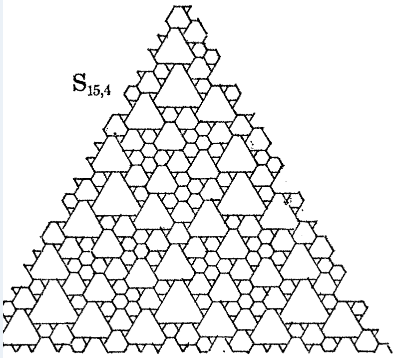


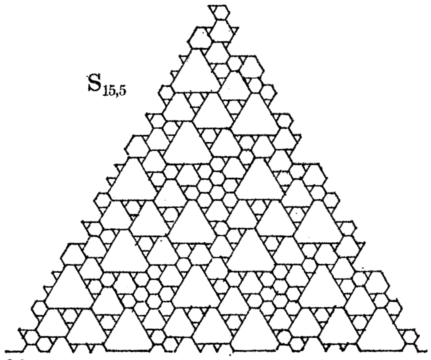


25

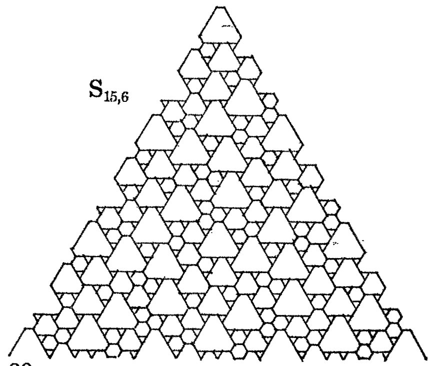


26

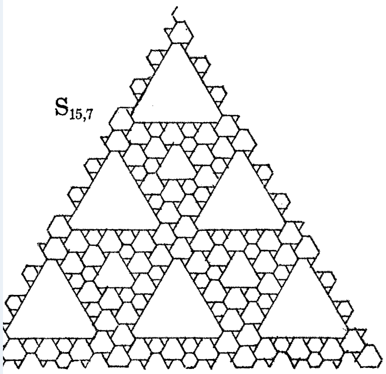


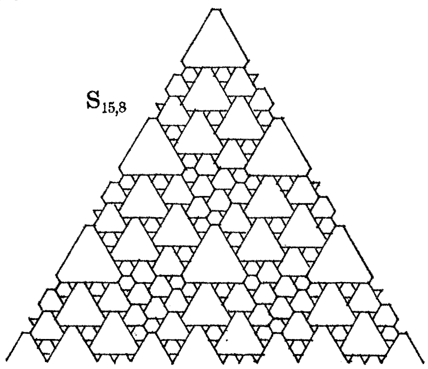


28

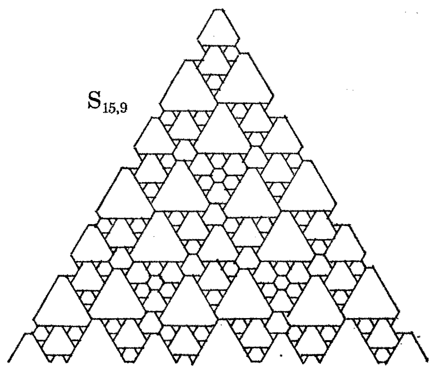


29

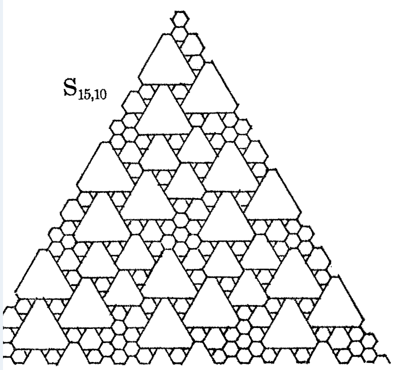


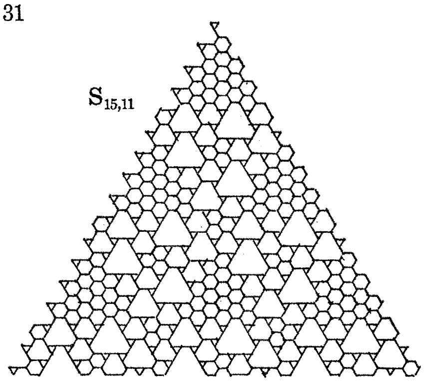


31

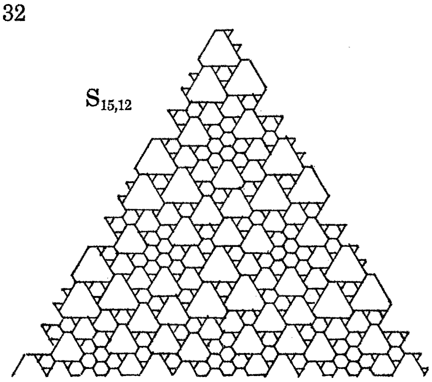


32



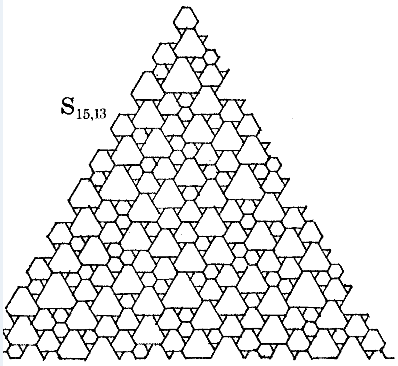


34

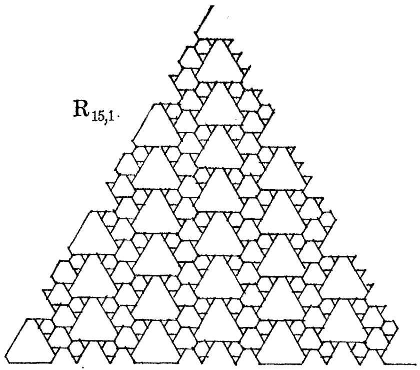


35

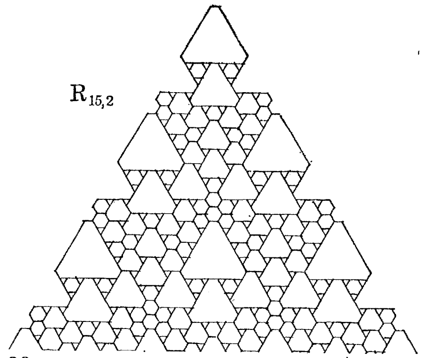
FIGURES 24–35. $S_{15,\,1(1)12}$.



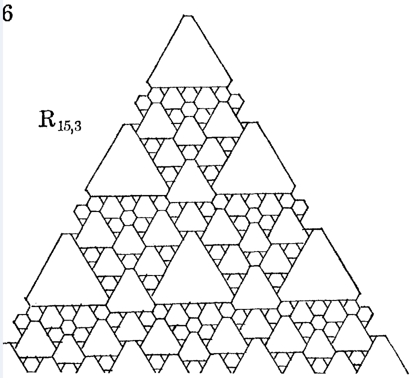
6



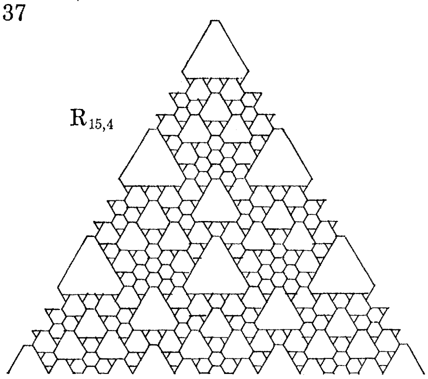
37



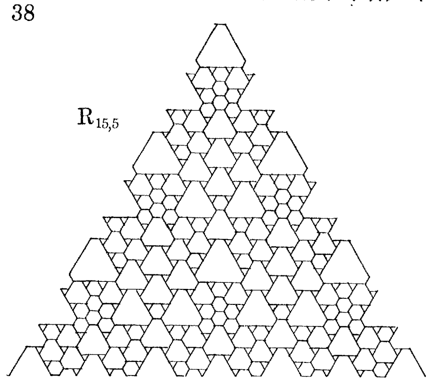
38



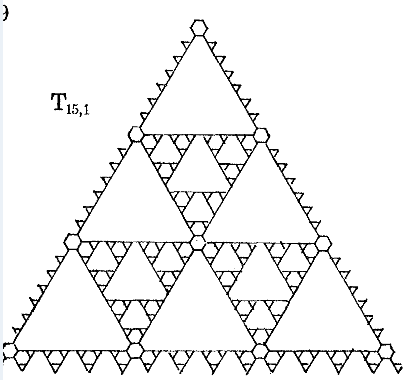
9



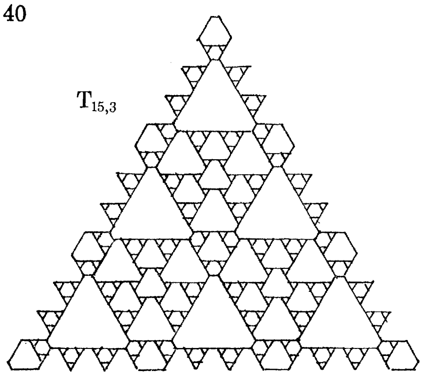
40



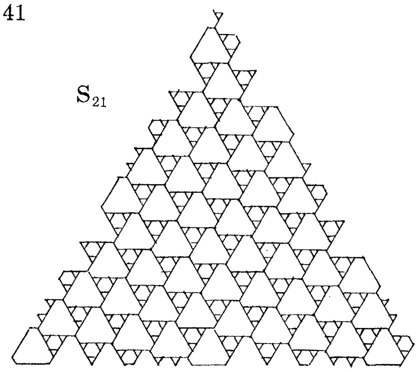
41



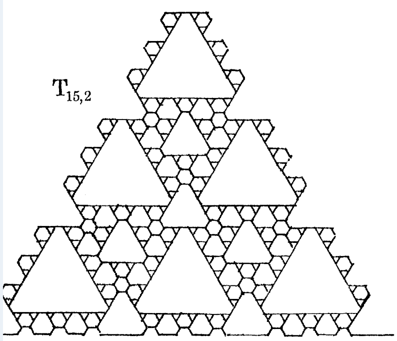
42

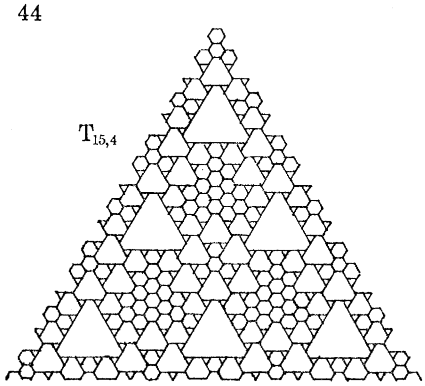


44

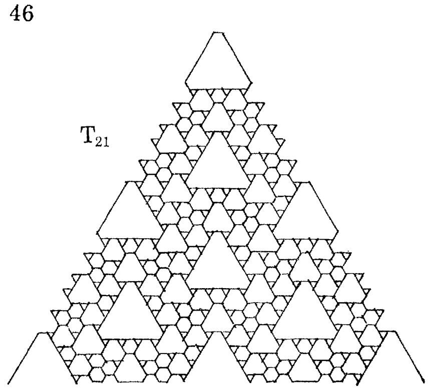


46





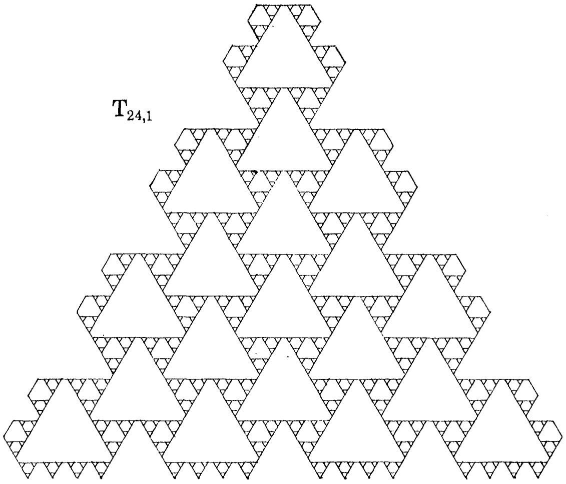
45



47

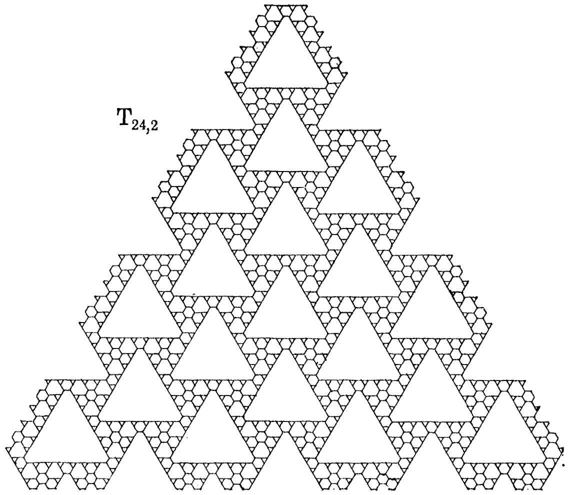
FIGURES 36–47. $S_{15,13}$, $R_{15,1(1)5}$, $T_{15,1(1)4}$, S_{21}, T_{21}.

$T_{24,1}$



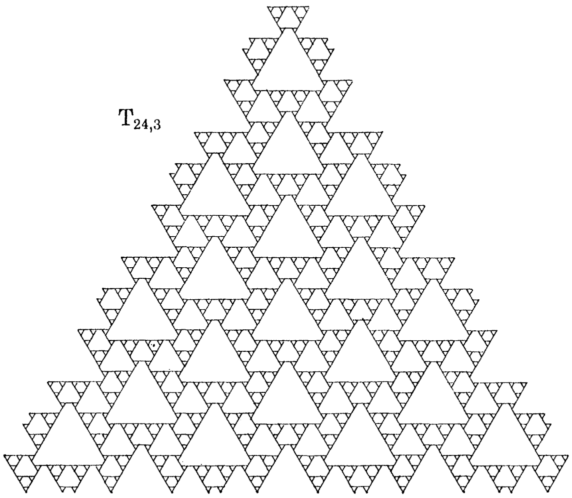
48

$T_{24,2}$



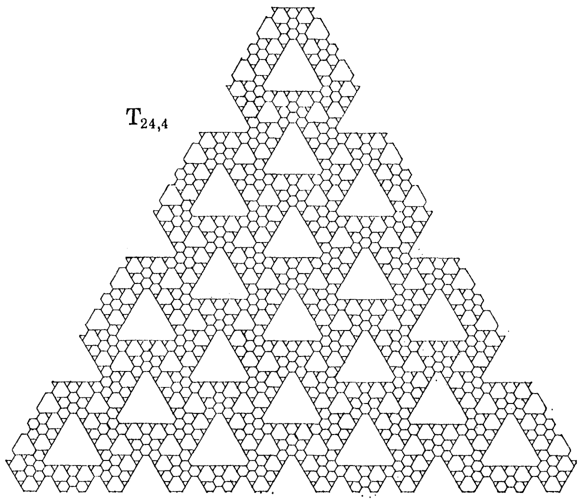
49

$T_{24,3}$



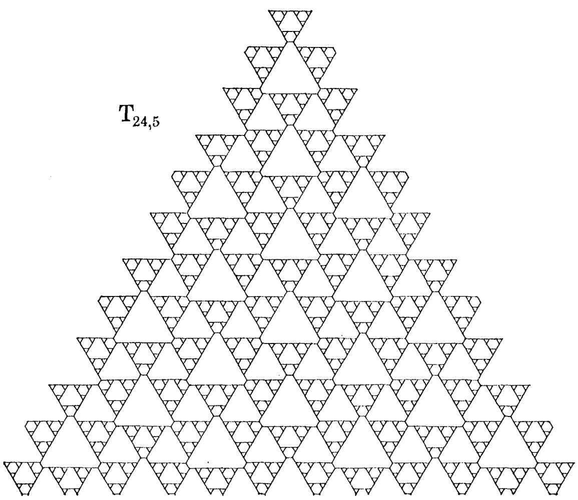
50

$T_{24,4}$



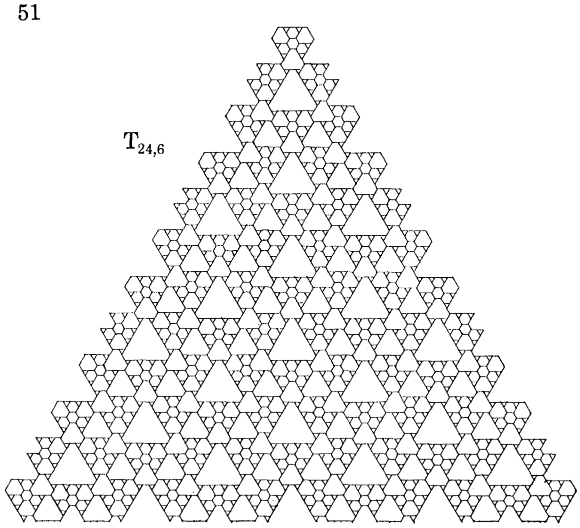
51

$T_{24,5}$



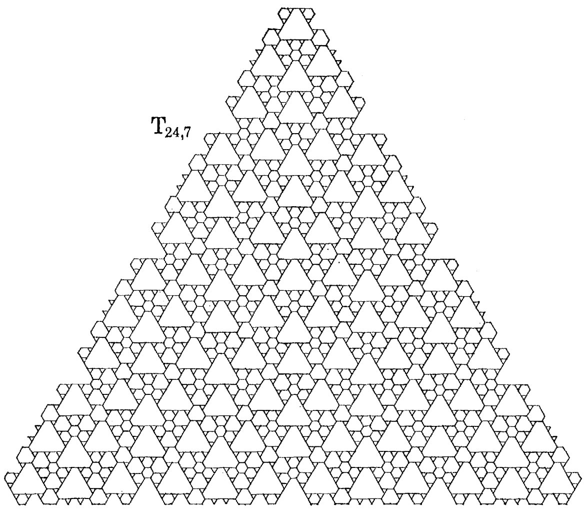
52

$T_{24,6}$

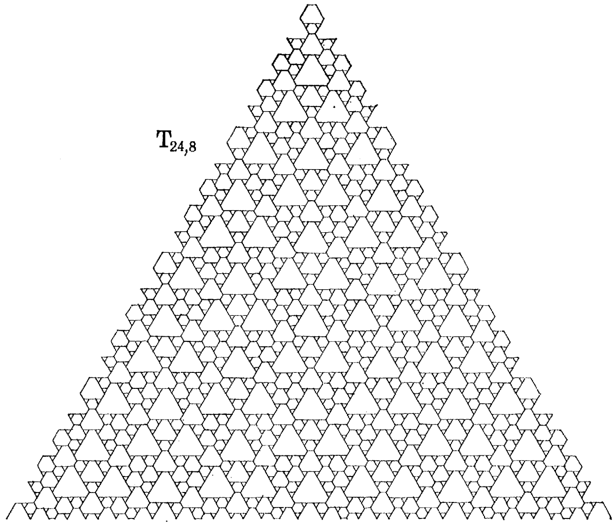


53

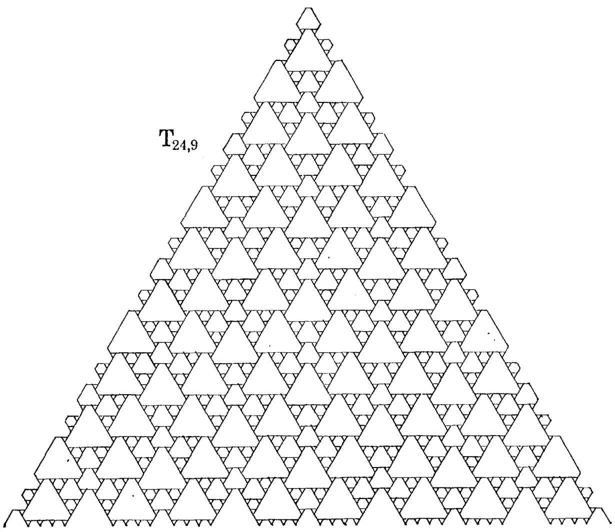
FIGURES 48–53. $T_{24,\,1(1)\,6}$.



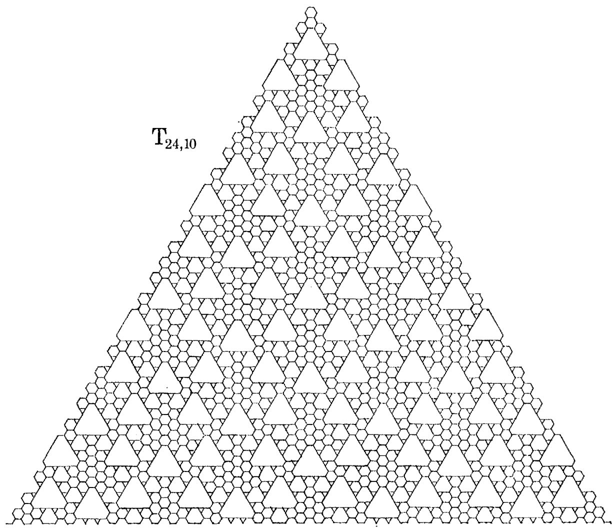
54



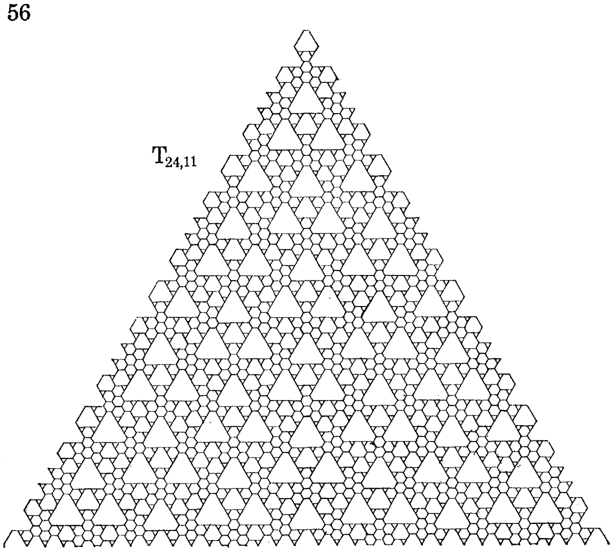
55



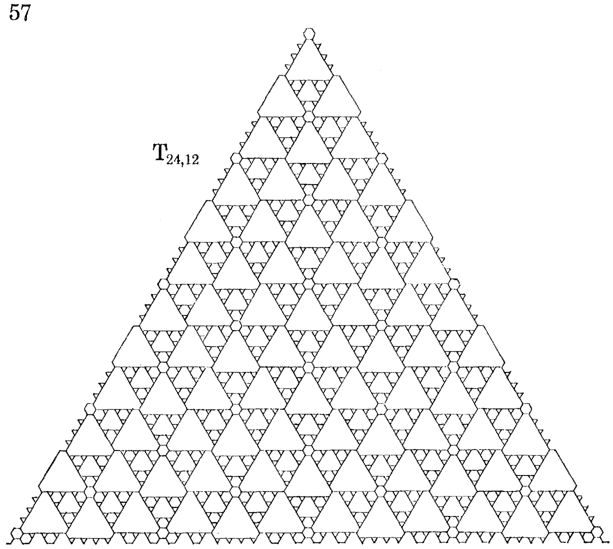
56



57

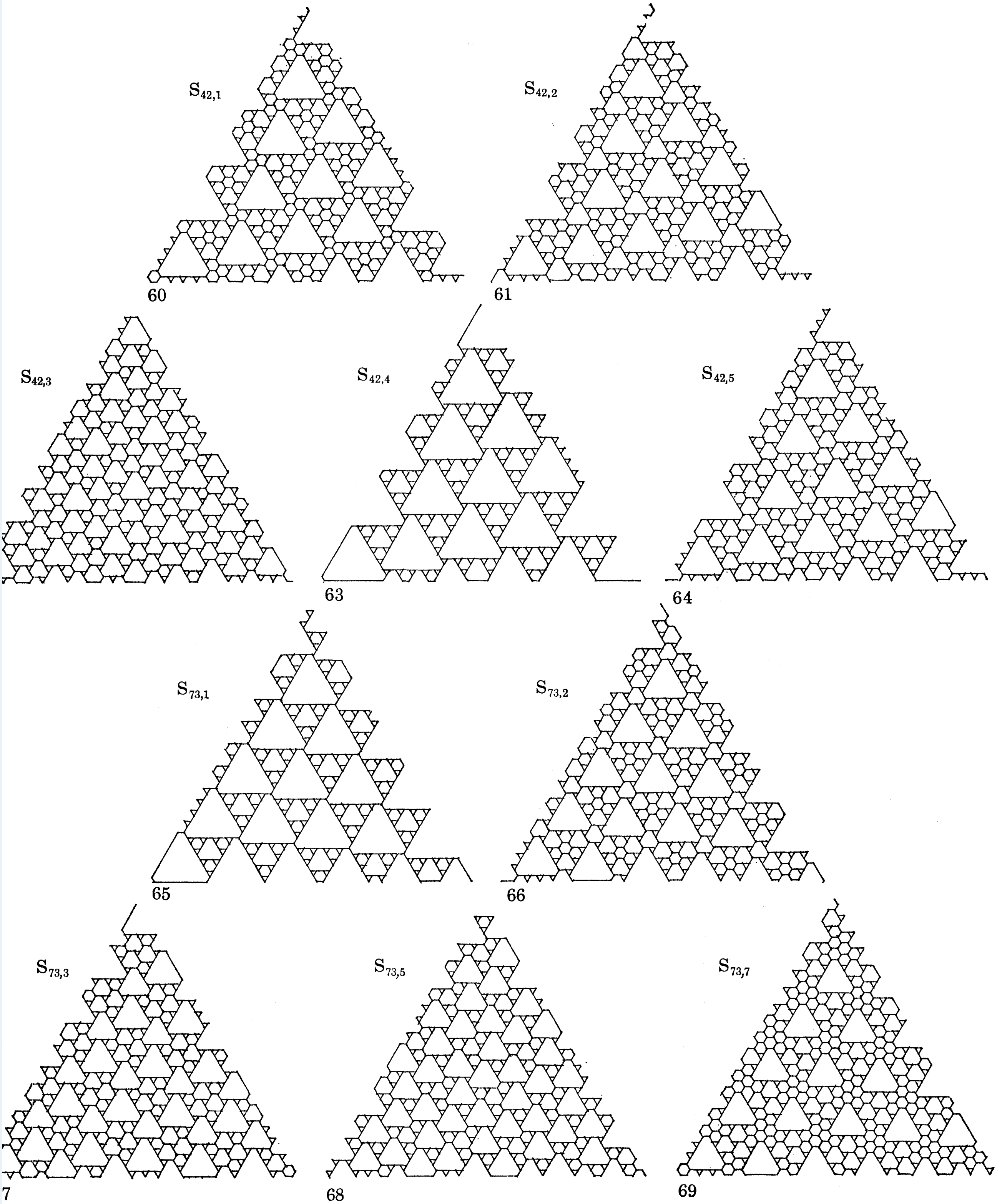


58

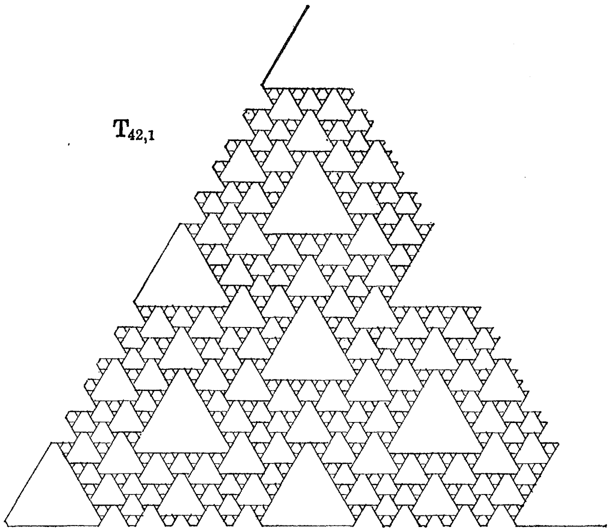


59

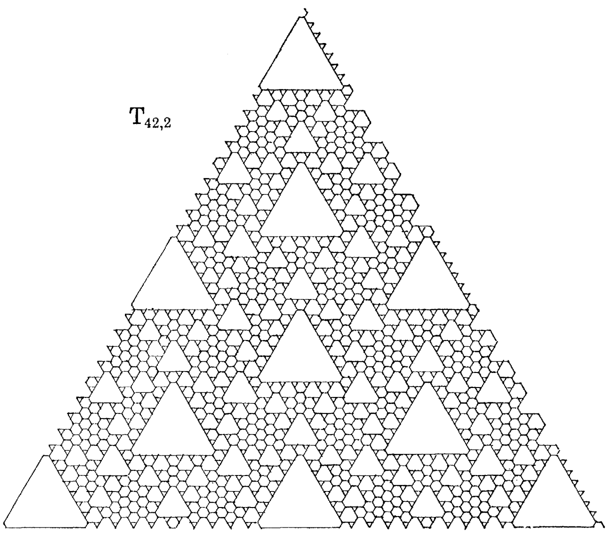
FIGURES 54–59. $T_{24,\ 7(1)\ 12}$.



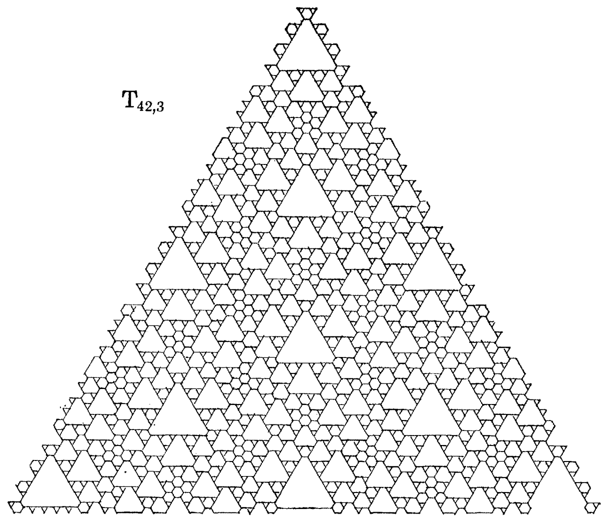
FIGURES 60–69. $S_{42,\,1(1)\,5}$, $S_{73,\,1(1)\,3,\,5,\,7}$.



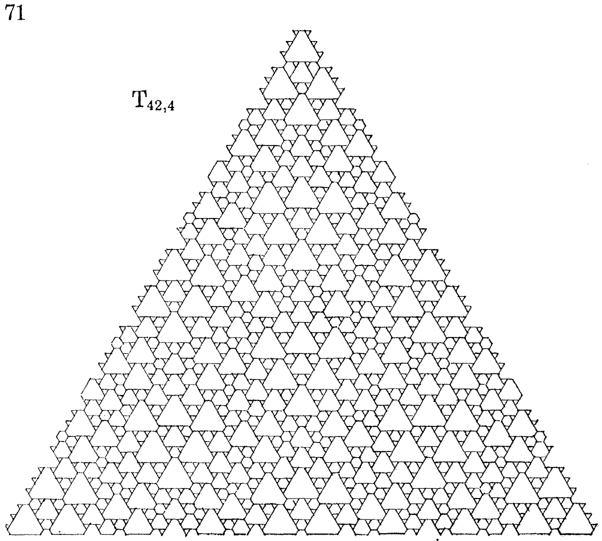
70



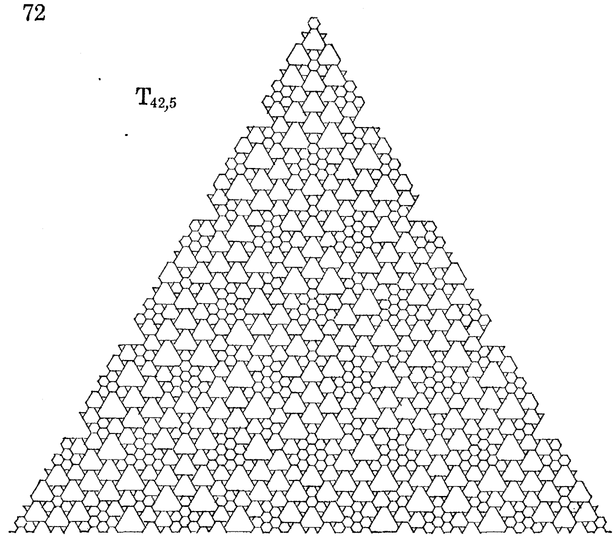
71



72



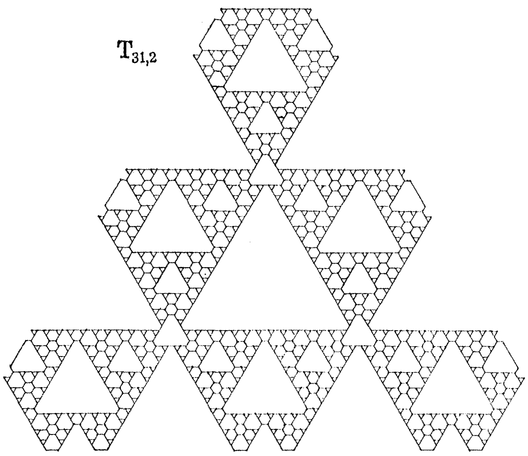
73



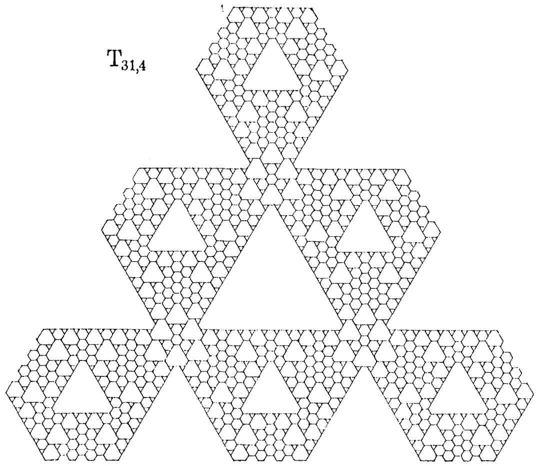
74

FIGURES 70–74. $T_{42,\,1(1)5}$.

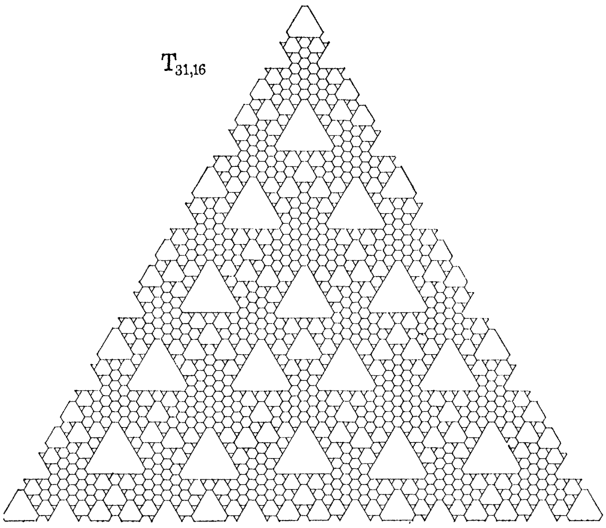
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF



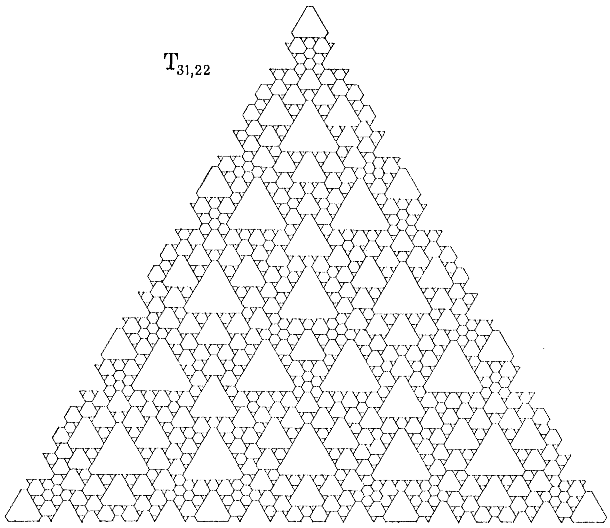
75



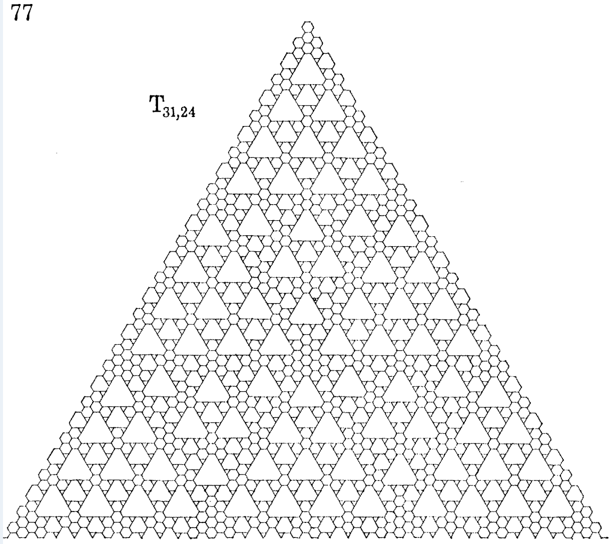
76



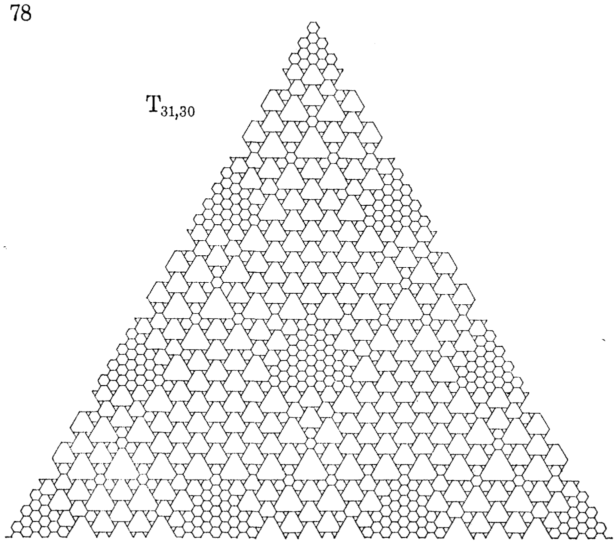
77



78



79



80

FIGURES 75–80. T_{31}.

TABLE 8. CONSTRUCTION OF FIGURES 17–80

figure no.		m	C	
17	T_3 (F1)	1	3	$1/b$
18	S_7 (F4)	1	7	$1/c$
19	R_5 (F2)	3	15	1/11111
20	T_6 (F3)	2	12	$1/b^2$
21	T_7 (F5)	7	49	$1/cc'$
22	$T_{12,3}$ (F17)	4	48	$11111/b^4$
23	U_{31}	1	31	1/110111
24	$S_{15,1}$ (F48)	15	225	$1011/C_{12}$
25	$S_{15,2}$ (F49)	15	225	$100101/C_{12}$
26	$S_{15,3}$ (F50)	15	225	$100011/C_{12}$
27	$S_{15,4}$ (F51)	15	225	$1011^2/C_{12}$
28	$S_{15,5}$ (F52)	15	225	$1000011/C_{12}$
29	$S_{15,6}$ (F53)	15	225	$10011101/C_{12}$
30	$S_{15,7}$ (F59)	15	225	$1011/bC_{12}$
31	$S_{15,8}$ (F66)	15	225	$1011^2/bC_{12}$
32	$S_{15,9}$ (F67)	15	225	$10000011/bC_{12}$
33	$S_{15,10}$ (F68)	15	225	$10001001/bC_{12}$
34	$S_{15,11}$ (F75)	15	225	$1001001011/bC_{12}$
35	$S_{15,12}$ (F76)	15	225	$1001101111/bC_{12}$
36	$S_{15,13}$ (F77)	15	225	$1011100111/bC_{12}$
37	$R_{15\ 1}$ (F36)	3	45	$1/b \times 11111$
38	$R_{15,2}$ (F38)	15	225	$1/10011 \times 11001$
39	$R_{15,3}$ (F40)	15	225	$1/b \times 10011 \times 11001$
40	$R_{15,4}$ (F55)	15	225	$1001001/bC_{12}$
41	$R_{15,5}$ (F57)	15	225	$111010111/bC_{12}$
42	$T_{15,1}$ (F54)	15	225	$1/bC_{12}$
43	$T_{15,2}$ (F45)	15	225	$1/C_{12}$
44	$T_{15,3}$ (F46)	15	225	b/C_{12}
45	$T_{15,4}$ (F47)	15	225	b^2/C_{12}
46	S_{21}	1	21	$1/bc$
47	T_{21}	7	147	$1/bcc'$
48	$T_{24,1}$	8	192	$1/b^8$
49	$T_{24,2}$	8	192	$1/b^7$
50	$T_{24,3}$	8	192	$1/b^6$
51	$T_{24,4}$	8	192	$1/b^5$
52	$T_{24,5}$	8	192	cc'/b^8
53	$T_{24,6}$	8	192	cc'/b^7
54	$T_{24,7}$	8	192	cc'/b^6
55	$T_{24,8}$	8	192	cc'/b^5
56	$T_{24,9}$	8	192	$(cc')^2/b^8$
57	$T_{24,10}$	8	192	$(cc')^2/b^7$
58	$T_{24,11}$	8	192	$(cc')^3/b^8$
59	$T_{24,12}$	8	192	$(cc')^3/b^7$
60	$S_{42,1}$	2	84	$1/bc^2$
61	$S_{12,2}$	2	84	$1/b^2c$
62	$S_{42,3}$	2	84	c'/bc^2
63	$S_{42,4}$	2	84	$1/b^2c^2$
64	$S_{42,5}$	2	84	c'/b^2c^2
65	$S_{73,1}$	1	73	$1/p_9$
66	$S_{73,2}$	1	73	b/p_9
67	$S_{73,3}$	1	73	b^2/p_9
68	$S_{73,5}$	1	73	b^4/p_9
69	$S_{73,7}$	1	73	b^6/p_9

Table 8

figure no.		m	C	
70	$T_{42,1}$	14	588	$1/(bcc')^2$
71	$T_{42,2}$	14	588	$1/b(cc')^2$
72	$T_{42,3}$	14	588	$1/b^2cc'$
73	$T_{42,4}$	14	588	$C_{12}/(bcc')^2$
74	$T_{42,5}$	14	588	$C_{12}/b(cc')^2$
75	$T_{31,2}$	31	961	$b/F_{31,*}F'_{31,*}$
76	$T_{31,4}$	31	961	$b^3/F_{31,*}F'_{31,*}$
77	$T_{31,16}$	31	961	$b^{15}/F_{31,*}F'_{31,*}$
78	$T_{31,22}$	31	961	$b^{21}/F_{31,*}F'_{31,*}$
79	$T_{31,24}$	31	961	$b^{23}/F_{31,*}F'_{31,*}$
80	$T_{31,30}$	31	961	$b^{29}/F_{31,*}F'_{31,*}$

In preparing this paper I am greatly indebted to Professor Ernst S. Selmer for his help with discussions and encouragement, and for his invitation to spend a sabbatical period at the University of Bergen where most of the writing up was done.

I have also received much help in the investigations from students in the University of Cambridge, in particular from Mr D. B. Webster, taking a Diploma in Statistics, who provided me with computer programs and obtained practical examples and tables for the study of the structure of binary sequences over GF(2). Also Mr J. J. B. Parker, taking a Diploma in Computing, provided programs and tables for the rotational and reflexive polynomials over GF(2) to higher degree than it is possible to give in this paper.

Finally I wish to thank Professor J. W. S. Cassels and the Department of Pure Mathematics and Mathematical Statistics for help provided, after my retirement in 1973, in accommodation and with the use of Xerox copying facilities, most helpful with the processing of diagrams.

References

ApSimon, H. G. 1970 Periodic forests whose largest clearings are of size 3. *Phil. Trans. R. Soc. Lond.* A **266**, 113–121.

ApSimon, H. G. 1970 Periodic forests whose largest clearings are of size $n \geqslant 4$. *Proc. R. Soc. Lond.* A **319**, 404.

Jacobi, C. G. J. 1839 *Canon arithmeticus.* Berolini, Typis Academicis.

Marsh, R. W. 1957 *Tables of irreducible polynomials over* GF(2). Washington, D.C.: National Security Agency.

Miller, J. C. P. 1968 Periodic forests of stunted trees. In *Computers in mathematical research* (ed. R. F. Churchhouse & J.-C. Herz), pp. 149–167. Amsterdam: North Holland Publishing Company.

Miller, J. C. P. 1970 Periodic forests of stunted trees. *Phil. Trans. R. Soc. Lond.* A **266**, 63–111.

Mossige, S. 1972 *Tables of irreducible binary polynomials* (*with polynomials in* GF(2)) *for degrees* 10(1)20. Prepared at the Mathematical Institute, University of Bergen, Norway.

Parker, J. J. B. 1972 *Rotational and reflexive polynomiala over* GF(2). Diploma dissertation, Computer Laboratory, University of Cambridge, England.

Peterson, W. W. 1961 *Error-correcting codes.* Cambridge, Mass.: M.I.T. Press.

Selmer, E. S. 1966 *Linear recurrence relations over finite fields.* Department of Mathematics, University of Bergen, Norway.

van Wijngaarden, A. 1966 *Triangular arrays of digits.* Amsterdam: Mathematisch Centrum.

Webster, D. B. 1967 *Applied field report*: Diploma in Statistics. Based on lecture course, 'Periodic Binary Sequences'. Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, England.

The colour plates in this issue have been printed by John Swain and Son Ltd., London.

Table 2. Table of bases for rotational vectors

$$\left.\begin{array}{l} 1^{2^j-3i-1}(t+1)^i\,t^{2^j-3i-1} \text{ of length } n = 2\cdot 2^j - 3i - 1 \\ 1^{2^k-2\lambda-2i-1}b^\lambda(t+1)^i\,t^{2^j-2\lambda-2i-1} \text{ of length } n = 2\cdot 2^j - 2\lambda - 3i - 1 \\ 1^{2^j-2\lambda-2i-4}b^\lambda c(t+1)^i\,t^{2^j-2\lambda-2i-4} \text{ of length } n = 2\cdot 2^j - 2\lambda - 3i - 4 \end{array}\right\} \quad \begin{array}{l} b = t^2+t+1, \quad c = t^3+t+1. \\ 0 \leqslant 4i \leqslant 2(2^j-k-1) \end{array}$$

i \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	1	.	1	c	b	.	1	b^2c	b^3	bc	b^2	c	b	.	1	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3
1				1			bc	b^2	c	b		1			b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c
2						c	b		1					b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1
3						1							b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1			
4												b^2c	b^3	bc	b^2	c	b		1						
5											bc	b^2	c	b		1									
6										c	b		1												
7										1															b^7c
8																								b^6c	b^7
9																							b^5c	b^6	b^4c
10																						b^4c	b^5	b^3c	b^4
11																					b^3c	b^4	b^2c	b^3	bc
12																				b^2c	b^3	bc	b^2	c	b
13																			bc	b^2	c	b		1	
14																		c	b		1				
15																		1							
16																									
17																									
18																									
19																									
20																									
21																									
22																									
23																									
24																									
25																									
26																									
27																									
28																									
29																									
30																									
31																									

i \ n	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
0	bc	b^2	c	b	.	1	$b^{14}c$	b^{15}	$b^{13}c$	b^{14}	$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c
1	b		1			$b^{13}c$	b^{14}	$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5
2					$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c
3				$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2
4			$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b	
5		b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1		
6	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1					
7	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1								
8	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1											
9	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1														
10	b^2c	b^3	bc	b^2	c	b		1																	
11	b^2	c	b		1																				
12		1																							
13																									
14																									$b^{16}c$
15																								$b^{15}c$	b^{16}
16																							$b^{14}c$	b^{15}	$b^{13}c$
17																						$b^{13}c$	b^{14}	$b^{12}c$	b^{13}
18																					$b^{12}c$	b^{13}	$b^{11}c$	b^{12}	$b^{10}c$
19																				$b^{11}c$	b^{12}	$b^{10}c$	b^{11}	b^9c	b^{10}
20																			$b^{10}c$	b^{11}	b^9c	b^{10}	b^8c	b^9	b^7c
21																		b^9c	b^{10}	b^8c	b^9	b^7c	b^8	b^6c	b^7
22																	b^8c	b^9	b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c
23																b^7c	b^8	b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4
24															b^6c	b^7	b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc
25														b^5c	b^6	b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b
26													b^4c	b^5	b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1	
27												b^3c	b^4	b^2c	b^3	bc	b^2	c	b		1				
28											b^2c	b^3	bc	b^2	c	b		1							
29										bc	b^2	c	b		1										
30									c	b		1													
31									1																

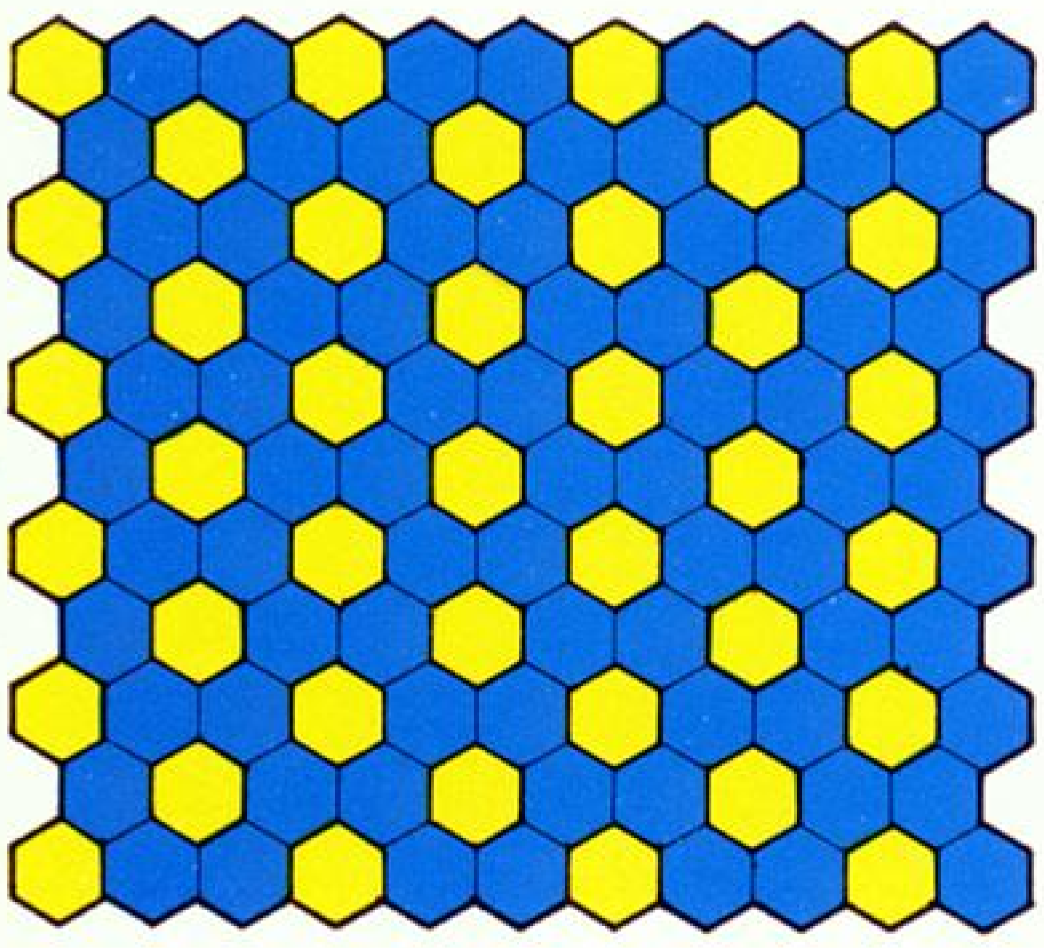
(Facing p. 610)

Figures 1 and 2 show all copses with $n = 1, 2$. For $n = 1$ these are the unit cells, corresponding to vacant (0, yellow) and live (1, blue) nodes. For $n = 2$ we have four arrangements of unit C-triangles, with two patterns under rotation and reflexion.

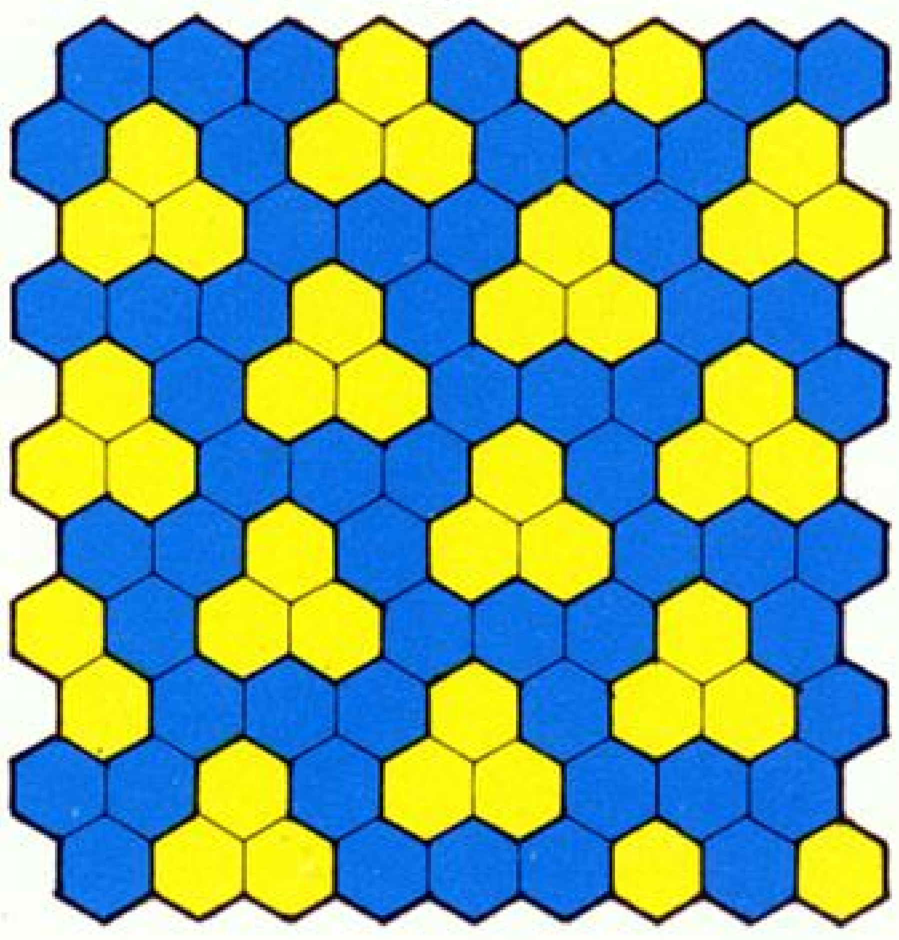
Figures 3 and 4 show all patterns for $n = 3$ and 4.

Figures 5–9 show all S- and T-patterns for $n = 5$ to 9.

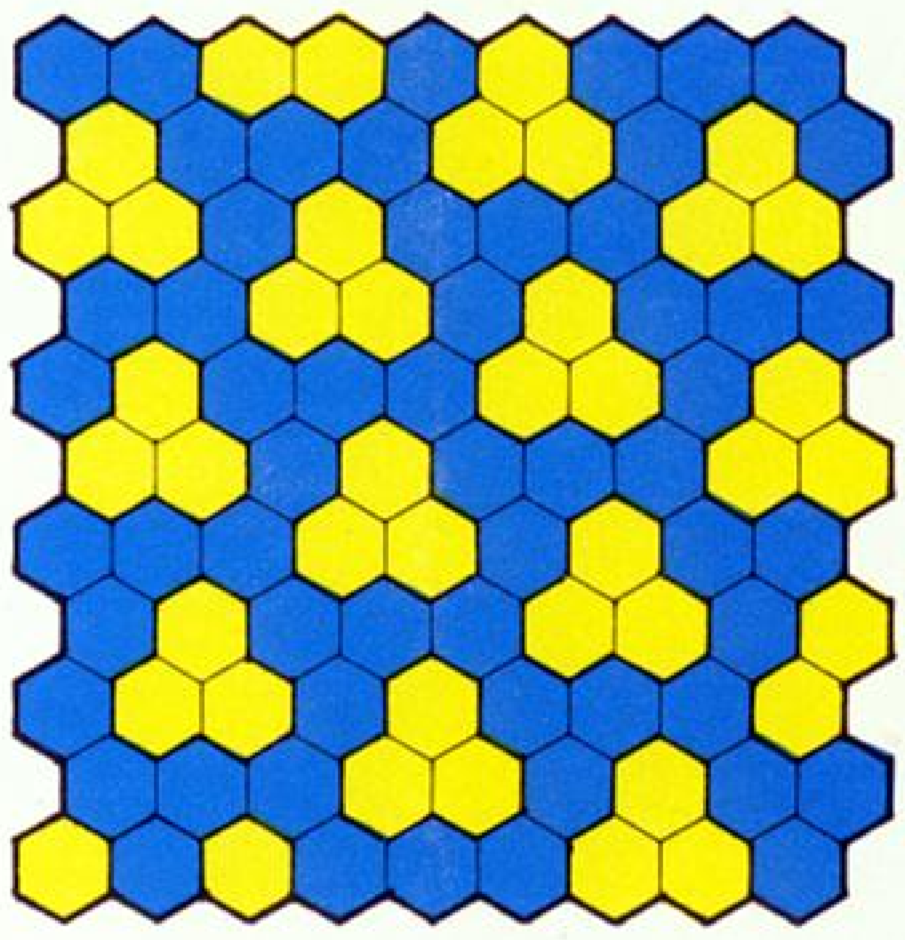
10



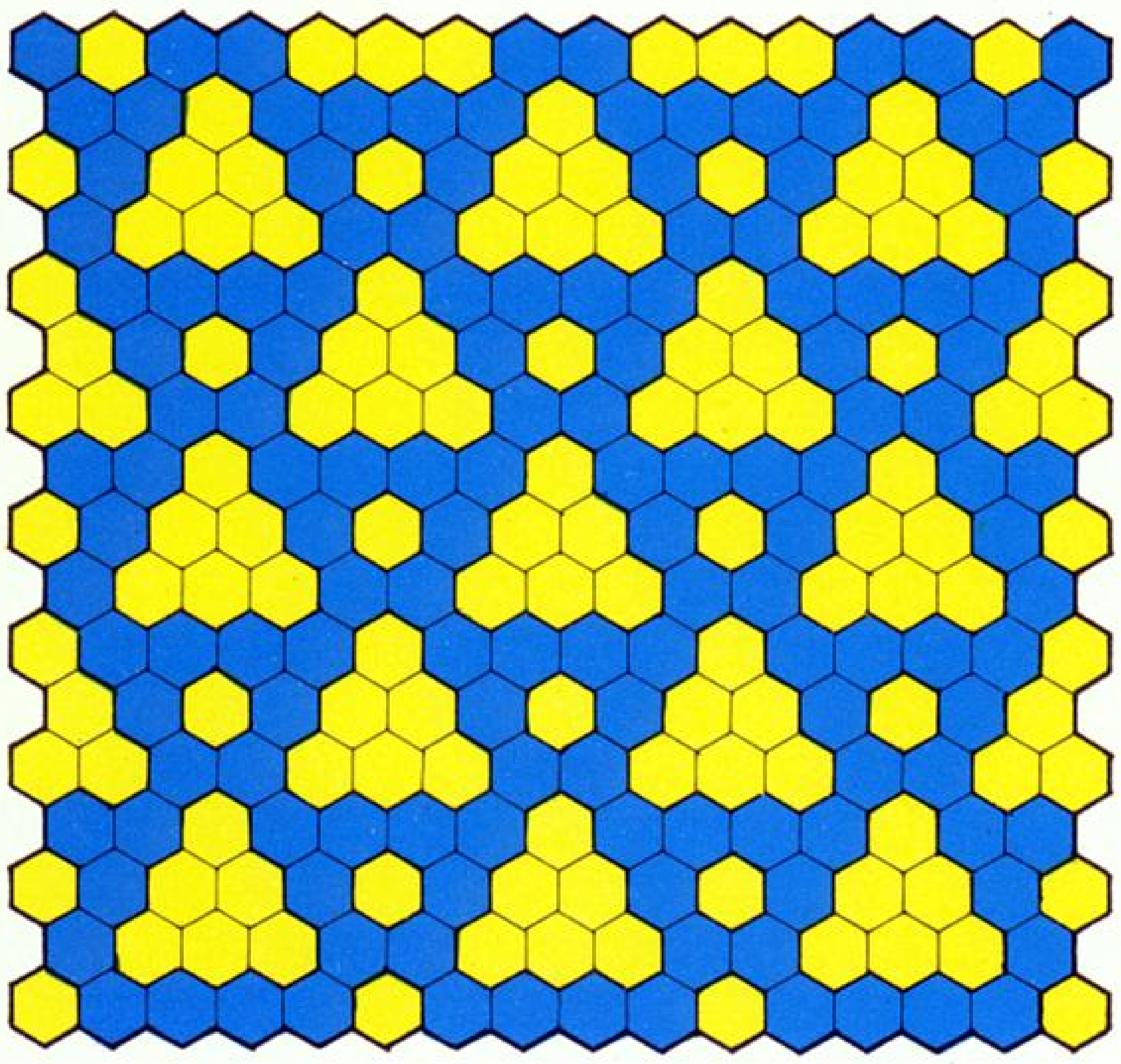
11



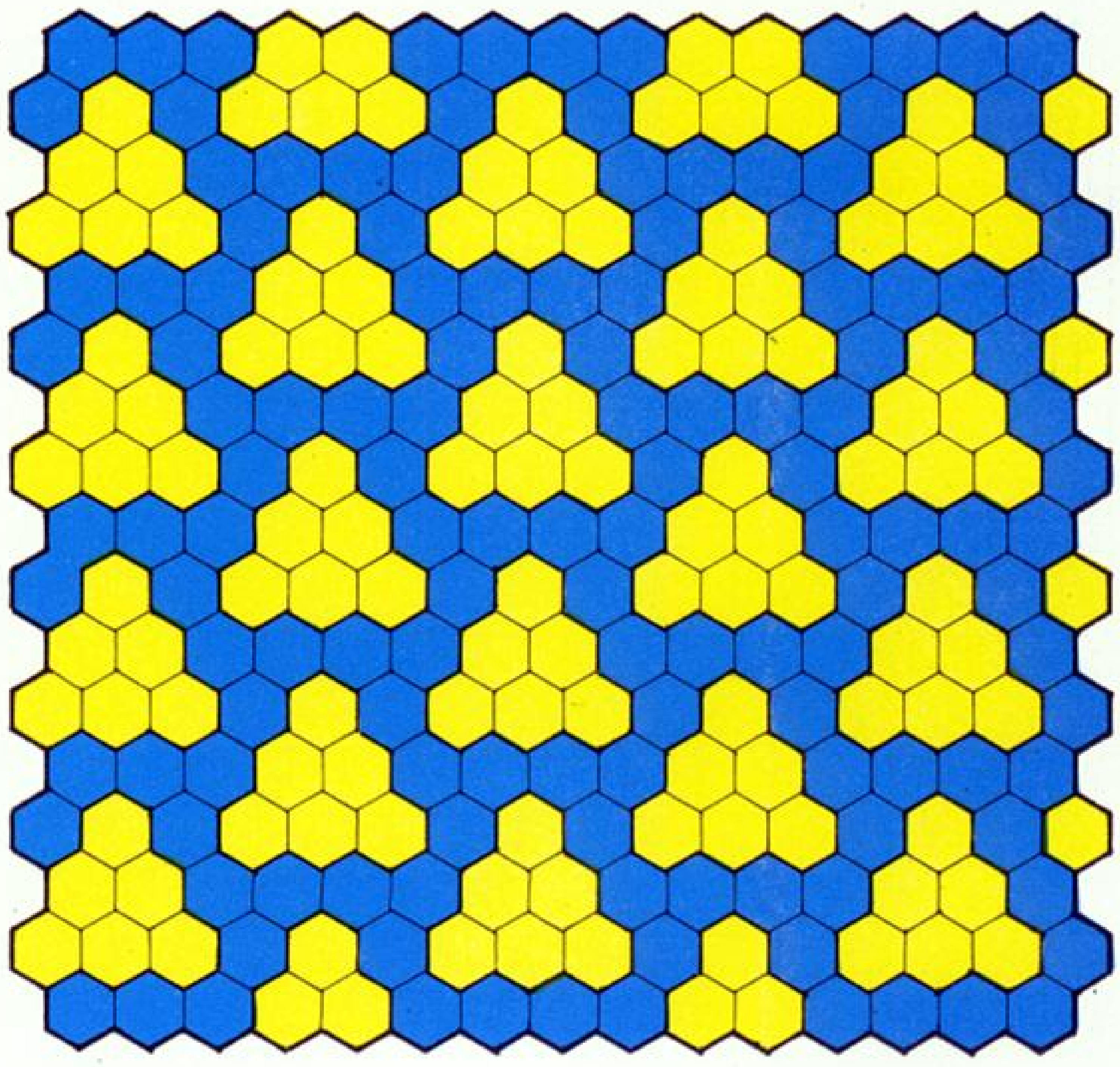
12



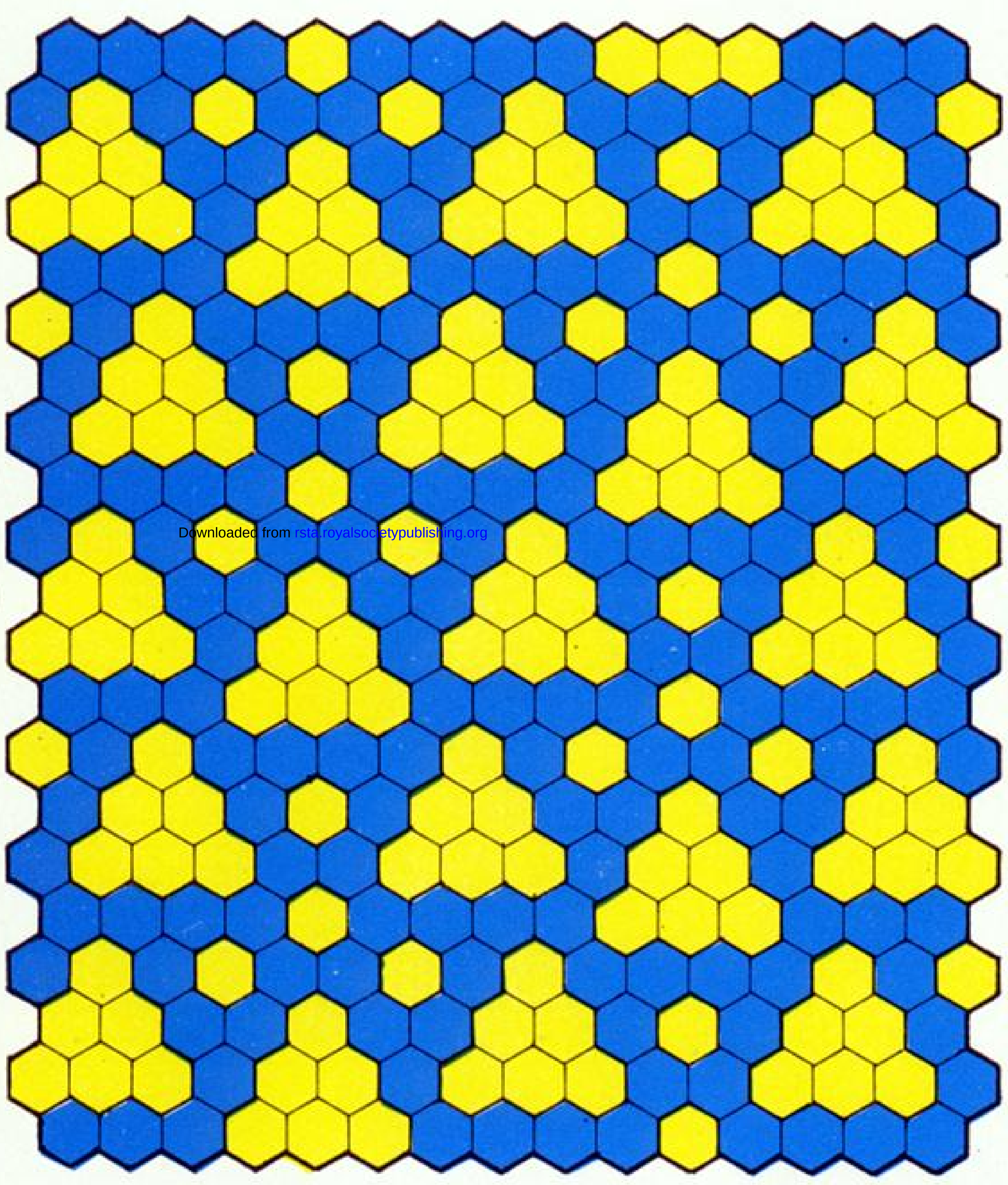
13



14



15



16

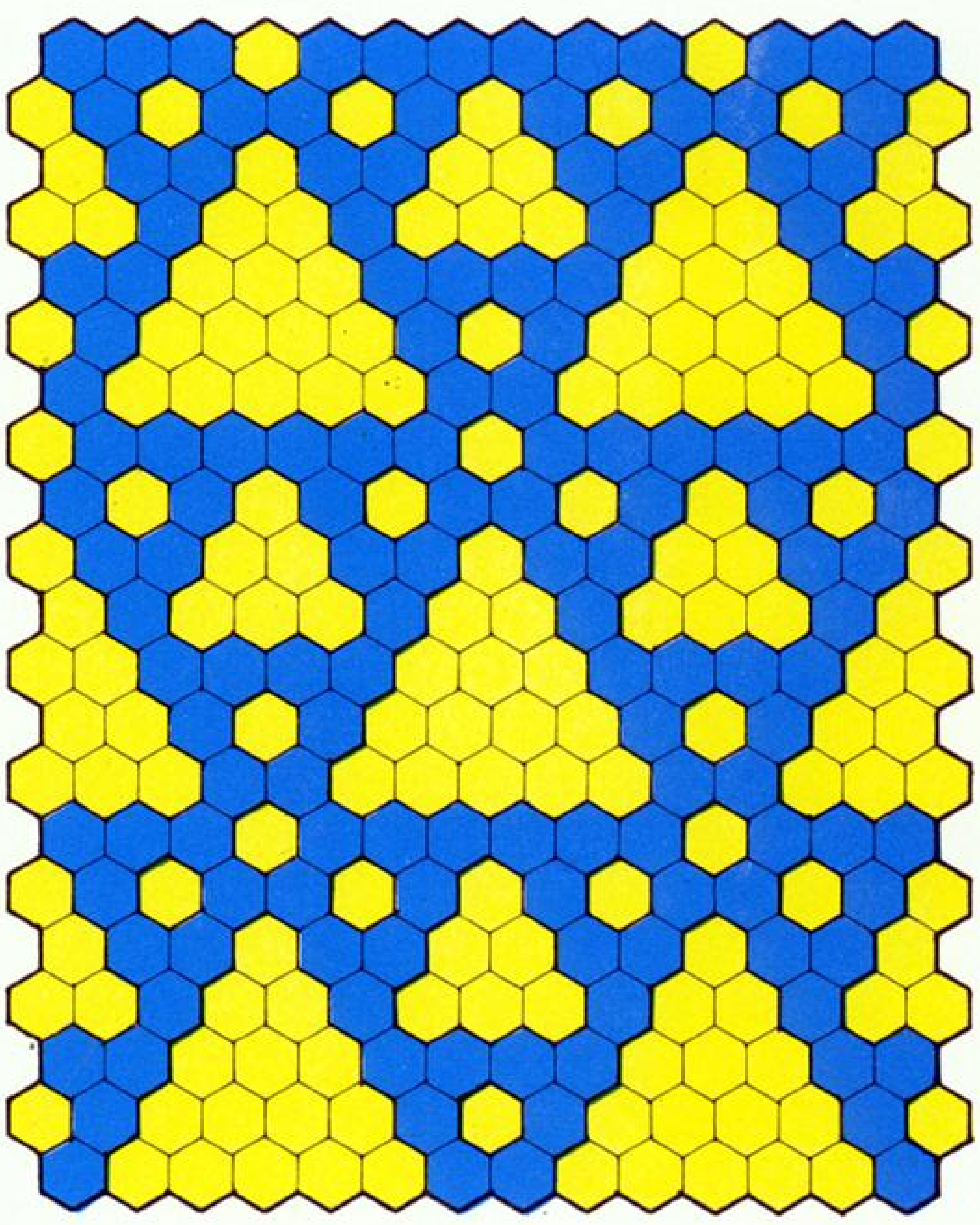


figure 10 shows T_3 (F1) 3×1

figure 11 shows S_7 (F4) 7×1

figure 12 shows as above, but mirror image

figure 13 shows R_5 (F2) 5×3

figure 14 shows T_6 (F3) 6×2

figure 15 shows $T_{12,3}$ (F17) 12×4

figure 16 shows T_7 (F5) 7×7

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES